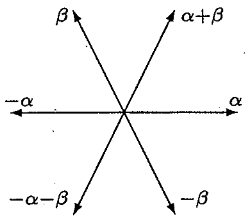
Lie Algebras IV 2008

Two-dimensional root systems

Example 11.6

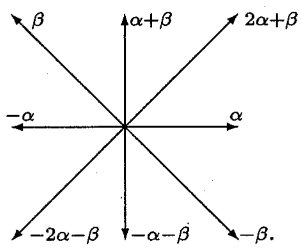
Let $E = \mathbf{R}^2$ with the Euclidean inner product. We shall find all root systems R contained in E. Take a root α of the shortest possible length. Since R spans E, it must contain some root $\beta \neq \pm\alpha$. By considering $-\beta$ if necessary, we may assume that β makes an obtuse angle with α. Moreover, we may assume that this angle, say θ, is as large as possible.

(a) Suppose that $\theta = 2\pi/3$. Using Proposition 11.5, we find that R contains the six roots shown below.



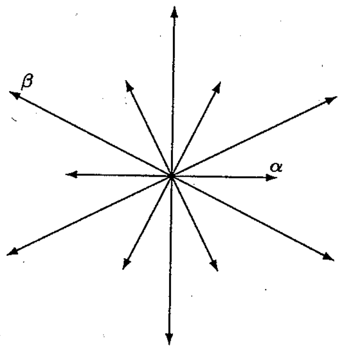
One can check that this set is closed under the action of the reflections $s_\alpha, s_\beta, s_{\alpha+\beta}$. As $s_{-\alpha} = s_\alpha$, and so on, this is sufficient to verify (R3). We have therefore found a root system in E. This root system is said to have *type* A_2. (The 2 refers to the dimension of the underlying space.)

(b) Suppose that $\theta = 3\pi/4$. Proposition 11.5 shows that $\alpha + \beta$ is a root, and applying s_α to β shows that $2\alpha + \beta$ is a root, so R must contain



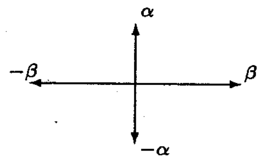
This root system is said to have *type* B_2. A further root would make an angle of at most $\pi/8$ with one of the existing roots, so this must be all of R.

(c) Suppose that $\theta = 5\pi/6$. We leave it to the reader to show that R must be



and to determine the correct labels for the remaining roots. This root system is said to have *type* G_2.

(d) Suppose that β is perpendicular to α. This gives us the root system of *type* $A_1 \times A_1$.



Here, as $(\alpha, \beta) = 0$, the reflection s_α fixes the roots $\pm\beta$ lying in the space perpendicular to α, so there is no interaction between the roots $\pm\alpha$ and $\pm\beta$. In particular, knowing the length of α tells us nothing about the length of β.

From 'Introduction to Lie Algebras'
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