Lie Algebras IV 2008

Root system of sl(n, C)

As usual let \( E_{ij} \) be the matrix with a 1 in the \((i, j)\) the position and 0's everywhere else. Let \( D_{ij} = E_{ii} - E_{jj} \). Let \( h_i = D_{ii+1} \). A basis for \( sl(n, \mathbb{C}) \) is given by the \( h_1, \ldots, h_{n-1} \) and the \( E_{ij} \) for \( i \neq j \). Let \( H \) be the subalgebra of diagonal (traceless) matrices which we have seen in lectures is a Cartan subalgebra. Let \( \epsilon_j : H \to \mathbb{C} \) be defined by \( \epsilon_j(\sum_{i=1}^n x_i E_{ii}) = x_j \). We have already seen that the roots are given by

\[
\Phi = \{ \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n \}
\]

and the \( \epsilon_i - \epsilon_j \) root space is spanned by \( E_{ij} \). Note that this makes sense as \( H \) has dimension \( n - 1 \) and \( sl(n, \mathbb{C}) \) has dimension \( n^2 - 1 \) so that we expect that there are \( n^2 - 1 - (n - 1) = n^2 - 1 \) roots. Note also that the \( \epsilon_i \) are not a basis of \( H^* \) as there are too many of them and \( \epsilon_1 + \cdots + \epsilon_n = 0 \). I leave it as an exercise to show that \( \epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n \) is a basis of \( H^* \).

If \( \alpha = \epsilon_i - \epsilon_j \) then the standard basis of \( sl(\alpha) \) is given by \( h_\alpha = D_{ij}, e_\alpha = E_{ij} \) and \( f_\alpha = E_{ji} \). This is easy to check as

\[
[h_\alpha, e_\alpha] = [D_{ij}, E_{ij}]
= [E_{ii} - E_{jj}, E_{ij}]
= [E_{ii}, E_{ij}] - [E_{ji}, E_{ij}]
= E_{ij} - (-1)E_{jj}
= 2E_{ij}
\]

Likewise \([h_\alpha, f_\alpha] = [D_{ij}, E_{ji}] = -[D_{ji}, E_{jj}] = -2E_{ji}\) and \([e_\alpha, f_\alpha] = [E_{ij}, E_{ji}] = E_{ii} - E_{jj} = D_{ij}\).

There is a theorem which didn’t prove that tells us the Killing form is a multiple of the form \( \text{tr}(XY) \) but we can compute directly when we just want the Killing form on \( H \). It suffices to compute \( \kappa(h_i, h_j) \). Notice that \( sl(n, \mathbb{C}) \) is an ideal in \( gl(n, \mathbb{C}) \) so that we can compute the Killing form in the latter space which makes things slightly easier. We have in \( gl(n, \mathbb{C}) \) that

\[
\kappa(h_i, h_j) = \kappa(E_{ii}, E_{jj}) - \kappa(E_{ii}, E_{j+1j+1}) - \kappa(E_{i+1i+1}, E_{jj}) + \kappa(E_{i+1i+1}, E_{j+1j+1}).
\]

so it suffices to compute \( \text{tr}(\text{ad}(E_{ii}) \text{ad}(E_{jj})) \) in \( gl(n, \mathbb{C}) \). We have

\[
\text{ad}(E_{ii}) \text{ad}(E_{jj})(E_{kl}) = (\delta_{ik} - \delta_{il})(\delta_{jk} - \delta_{lj})E_{kl}
\]

so that

\[
\text{tr}(\text{ad}(E_{ii}) \text{ad}(E_{jj})) = \sum_{kl} (\delta_{ik} - \delta_{il})(\delta_{jk} - \delta_{lj})
= \sum_{kl} (\delta_{ik}\delta_{jk} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk} + \delta_{il}\delta_{lj})
= n\delta_{ij} - 1 - n\delta_{ij}
= 2n(\delta_{ij} - 1).
\]

But we also have \( \text{tr}(E_{ii}E_{jj}) = \delta_{ij} \) so that \( \text{tr}(\text{ad}(E_{ii}) \text{ad}(E_{jj})) = 2n \text{tr}(E_{ii}E_{jj}) \) and it is easy to then show that

\[
\kappa(h_i, h_j) = 2n \text{tr}(h_i h_j).
\]

As we have remarked in lectures a root system is unchanged if we scale the inner product. So we may as well work with \((1/2n)\kappa\) which is more convenient. Notice that it is easy to identify \( H \) with a subspace of \( \mathbb{C}^n \) by mapping any diagonal matrix to the vector of its diagonal entries. With this choice the roots are the vectors \( \alpha_{ij} = \epsilon_i - \epsilon_j \) which we think of as living in \( \mathbb{R}^n \) and the inner product of two roots is just the usual inner product on \( \mathbb{R}^n \) restricted to the \( n - 1 \) dimensional subspace

\[
E = \{ x \in \mathbb{R}^n \mid x^1 + x^2 + \cdots + x^n = 0 \}.
\]

The roots are

\[
\Phi = \{ \epsilon_i - \epsilon_j \mid i \neq j \}
\]

if, as usual, \( \epsilon_i \) is the vector with a one in the \( i \)th place and zeros elsewhere. If we want to find a base for \( \Phi \) then we note that \((\epsilon_i - \epsilon_j)(z) = z_i - z_j \) so we need to choose a \( z \in E \) which has all its components distinct.
One choice is $z_1 > z_2 > \cdots > z_n$ and then the base of simple roots is given by $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n$. Notice that we have

\[
(\alpha_i, \alpha_j) = \begin{cases} 
2 & \text{if } |i - j| = 0 \\
-1 & \text{if } |i - j| = 1 \\
0 & \text{if } |i - j| > 1 
\end{cases}
\]

The Cartan matrix is therefore

\[
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2 \\
\end{pmatrix}
\]

and the Dynkin diagram is

\[
\begin{array}{c}
\circ \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\
\end{array}
\]

Finally consider $\mathfrak{sl}(3, \mathbb{C})$ where we can draw the root system. We take as a base of simple roots the roots $\alpha_1 = e_1 - e_2 = (1, -1, 0)$ and $\alpha_2 = e_2 - e_3 = (0, 1, -1)$. We have $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$ and $(\alpha_1, \alpha_2) = -1$. Hence the angle between $\alpha_1$ and $\alpha_2$ is $2\pi/3$ and we have the root system $A_2$:

\[
\alpha_2 = (0, 1, -1) \quad \alpha_1 + \alpha_2 = (1, 0, -1) \\
-\alpha_1 = (-1, 1, 0) \quad \alpha_1 = (1, -1, 0) \\
-\alpha_1 - \alpha_2 = (-1, 0, 1) \quad -\alpha_2 = (0, -1, 1)
\]