Lie Algebras IV 2008

Conventions for matrices

If $X: V \to V$ is a linear map on a finite dimensional vector space V and V has a basis $v^1, \dots, v^n$ then we can find a matrix $[X]$ which represents X. But what is it? We would like to have that if $Y: V \to V$ also then $(YX): V \to V$ has matrix $[YX] = [Y][X]$. In fancy language we want the map

$$gl(V) \to gl(n, \mathbb{C})$$
$$X \mapsto [X]$$

to be an algebra isomorphism.

To obtain this we need to define $[X]$ by

$$X(v^i) = \sum_{j=1}^{n} X_{ji} v^j$$

which is not the obvious thing you might write down. But it works because

$$\begin{aligned}
\sum_{k=1}^{n} [YX]_{ki} v^k &= (YX)(v^i) \\
&= Y(X(v^i)) \\
&= Y(\sum_{j=1}^{n} X_{ji} v^j) \\
&= \sum_{j=1}^{n} X_{ji} Y(v^j) \\
&= \sum_{j=1}^{n} X_{ji} \left(\sum_{k=1}^{n} Y_{kj} v^k \right) \\
&= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} Y_{kj} X_{ji} \right) v^k \\
&= \sum_{k=1}^{n} ([Y][X])_{ki} v^k
\end{aligned}$$

As another test this is a good idea consider the action of a matrix X on column vectors. This defines a linear map $X: \mathbb{C}^n \to \mathbb{C}^n$. We would rather hope that if we expand this linear map in the usual basis $\{e^1, \dots, e^n\}$ using the definition above that we find that the ijth element is X_{ij}. To see this note that if I apply the matrix X to the vector e^i in the standard way it picks out the ith column of X which is also $\sum_{j=1}^{n} X_{ji} e^j$. So $X(e^i) = \sum_{j=1}^{n} X_{ji} e^j$ as we wanted.