Lie Algebras IV 2008

Killing form examples.

1. Killing form of $sl(2,\mathbb{C})$

Recall from the handout about $sl(2,\mathbb{C})$ representations that the matrices for the adjoint representation are:

$$\mathrm{ad}(e) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{ad}(f) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{ad}(h) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

From this we can calculate that

$$\kappa(h,h) = 8, \kappa(e,e) = \mathrm{tr}\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \quad \text{and} \quad \kappa(f,f) = \mathrm{tr}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = 0$$

and

$$\kappa(e,f) = \mathrm{tr}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 4, \kappa(e,h) = \mathrm{tr}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} = 0, \quad \text{and} \quad \kappa(f,h) = \mathrm{tr}\begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

We check non-degeneracy. Assume that $x = \alpha e + \beta h + \gamma f$ is in L^o that is $\kappa(x,y) = 0$ for all $y \in L$. Then $0 = \kappa(x,e) = 4\gamma$, $0 = \kappa(x,h) = 8\alpha$ and $0 = \kappa(x,f) = 4\alpha$ so that $x = 0$.

2. Killing form of Heisenberg Lie algebra

Recall that we have

$$f = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $[f,g] = z \in Z(L)$ and all other commutators being zero. Taking $\{f,z,g\}$ as a basis of L we obtain

$$\mathrm{ad}(f) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathrm{ad}(z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{ad}(g) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $\kappa(z,x) = 0$ for any $x \in L$ and, moreover, $\kappa(x,y) = 0$ for any $x,y \in L$. Hence $L^0 = L$ and we deduce the fact that L is solvable. We know this because $[L,L] = z\mathbb{C}$ and hence $L^{(2)} = 0$.