Lie Algebras IV 2008

Non-degenerate bilinear forms

I want to show that if V is a finite dimensional vector space with a non-degenerate, symmmetric, bilinear form $(\ ,\ )$ and $W \subset V$ then

$$\dim(W) + \dim(W^\perp) = \dim(V)$$

First we prove a result that doesn't need the bilinear form. Recall that V^*, the dual space of V is the vector space of all linear maps $\alpha\colon V \to \mathbb{F}$. If $v^1, \dots, v^n$ is a basis of V then a linear map $\alpha\colon V^* \to \mathbb{F}$ is determined by the n numbers $\alpha(v^1), \dots, \alpha(v^n)$ and any collection of n numbers $\alpha_1, \dots, \alpha_n$ determines a linear map α by $\alpha(\sum_{i=1}^n x^i v^i) = \sum_{i=1}^n \alpha_i x^i$. It follows that we can define the so-called dual basis $\xi^1, \dots, \xi^n$ of V^* by requiring that $\xi^j(v^i) = \delta_{ij}$ for all $i, j = 1, \dots, n$.

Exercise 0.1. Prove that $\xi^1, \dots, \xi^n$ is a basis of V^*.

If $W \subset V$ we define $W^\circ \subset V^*$ by

$$W^\circ = \{\alpha \in V \mid \alpha(w) = 0 \forall w \in W\}.$$

We have

Proposition 0.1.

$$\dim(W) + \dim(W^\circ) = \dim(V)$$

Proof. Let $v^1, \dots, v^r$ be a basis for W and extend it to a basis $v^1, \dots, v^n$ of V. Consider the dual basis $\xi^1, \dots, \xi^n$. We have that $\xi^{r+1}, \dots, \xi^n$ are linearly independent and in W° because they are zero when applied to the basis elements of W and hence when applied to any vector in W. We show that they span W°. Let α be in W° and write $\alpha = \sum_{i=1}^n \alpha_i \xi^i$. We have that $\alpha(v^j) = \sum_{i=1}^n \alpha_i \xi^i(v^j) = \alpha_j$. Hence $\alpha_1 = \cdots = \alpha_r = 0$ so that $\alpha = \sum_{i=r+1}^n \alpha_i \xi^i$ as required. □

Returning to a symmetric, bilinear form we define a map $\phi\colon V \to V^*$ by $\phi(v)(w) = (v, w)$.

Exercise 0.2. Show that ϕ is a linear map and $\ker(\phi) = V^\circ$.

It follows that if the form is non-degenerate then $\ker(\phi) = 0$ and that ϕ is an isomorphism on dimension grounds.

Proposition 0.2. *If* $W \subset V$ *then* $\phi(W^\perp) = W^\circ$. *So that* $\dim(W) + \dim(W^\perp) = \dim(V)$.

Proof. Exercise. □