Lie Algebras IV 2008

Assignment 3. Due Thursday 15th May.

1. Let W be a subspace of a finite-dimensional vector space V. Let $\{w^1, \dots, w^p\}$ be a basis of W. If $v^1, \dots, v^q$ are vectors in V show that $\{w^1, \dots, w^p, v^1, \dots, v^q\}$ is a basis of V if and only if $\{v^1 + W, \dots, v^q + W\}$ is a basis of V/W.

2. Let V be a finite dimensional, complex, vector space and $x \in gl(V)$. Assume that $0 \neq v \in V$ is in the kernel of x and let $\mathbb{C}v$ be the span of v in V. Show that $\bar{x}\colon V/\mathbb{C}v \to V/\mathbb{C}v$, defined by $\bar{x}(w + \mathbb{C}v) = x(w) + \mathbb{C}v$ is well-defined. If $\{v^1 + \mathbb{C}v, \dots, v^r + \mathbb{C}v\}$ is a basis of $V/\mathbb{C}v$ with respect to which $\bar{x}$ is strictly upper triangular show that $\{v, v^1, \dots, v^r\}$ is a basis of V with respect to which x is strictly upper triangular.

3. Let $L = b(n, \mathbb{C})$ be the Lie algebra of all $n \times n$ upper triangular matrices and for $i = 1, \dots, n$ let $e^i \in \mathbb{C}^n$ be the vector with a 1 in the ith entry and zeroes elsewhere. For each $k = 1, \dots, n$ let W_k be the span of $\{e^1, \dots, e^k\}$.

a) Show that W_k is a submodule of the L-module $\mathbb{C}^n$.

b) If $x \in L$ what is the matrix for the action of x on W_k with respect to the basis $\{e^1, \dots, e^k\}$?

c) If $x \in L$ what is the matrix for the action of $\bar{x}$ on $\mathbb{C}^n/W_k$ with respect to the basis $\{e^{k+1} + W_k, \dots, e^n + W_k\}$?

4. Let L be a Lie algebra and let $\theta\colon V \to W$ be an L-module homomorphism between L-modules V and W. Show that $\bar{\theta}\colon V/\ker(\theta) \to \operatorname{im}(\theta)$ is an L-module isomorphism where $\bar{\theta}(v + \ker(\theta)) = \theta(v)$.

5. Let A be a subalgebra of a Lie algebra L and define the *normaliser* of A in L by

$$N_L(A) = \{x \in L \mid [x, a] \in A \ \forall a \in A\}.$$

a) Show that $N_L(A)$ is a subalgebra of L.

b) Show that $A \subset N_L(A)$ and, moreover, A is an ideal in $N_L(A)$.

c) Show that $N_L(A)$ is the largest subalgebra of L containing A in which A is an ideal.