# Lie Algebras IV 2008

## Assignment 2. Due Thursday 10th April.

1. Let the Lie algebra $L = L_1 \oplus L_2$ be a direct sum of two arbitrary Lie algebras $L_1$ and $L_2$. (This is not necessarily the $L_2$ defined in class.)

   a) Show that $Z(L) = Z(L_1) \oplus Z(L_2)$.

   b) Show that $L' = L_1' \oplus L_2'$.

   c) Prove that $L$ is solvable if and only if $L_1$ and $L_2$ are solvable.

2. Let $L$ be a Lie algebra. Show that $L$ has a non-zero solvable ideal if and only if $L$ has a non-zero abelian ideal.

3. Let $L$ be a non-abelian Lie algebra. Show that $\dim(Z(L)) \leq \dim(L) - 2$.

4. Let $L$ be a two-dimensional vector space with basis $x$ and $y$. Show that requiring bilinearity and antisymmetry and defining the bracket of $x$ and $y$ by $[x, y] = x$ suffices to define a unique Lie algebra $L$. (Hint: recall that is enough to check the Jacobi identity on three basis vectors).

5. Let $L = \{X + iY \mid X, Y \in su(2, \mathbb{C})\}$. Show that $L$ is a three-dimensional, complex, subalgebra of $gl(2, \mathbb{C})$. Which complex Lie algebra is it ?

6. Consider the Lie algebra $sl(2, \mathbb{C})$ with basis $h$, $e$ and $f$ satisfying $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$.

   a) Show that if $I$ is an ideal in $sl(2, \mathbb{C})$ and $h \in I$ then $I = sl(2, \mathbb{C})$.

   b) Show that $sl(2, \mathbb{C})$ has no non-trivial ideas. (Hint: Let $x = \alpha h + \beta e + \gamma f \in I$ and try to use the fact that $[e, x]$, $[f, x]$ and $[h, x]$ are in $I$ to show that $h \in I$. )

   c) Deduce from (b) that $sl(2, \mathbb{C})$ is semisimple.