Lie Algebras IV 2008

Assignment 1. Due Tuesday 18th March.

1. Show that $\mathbb{R}^3$ with the cross (vector) product as bracket is a Lie algebra. Calculate its centre and derived algebra.

2. Consider the Lie algebra $gl_n(\mathbb{C})$.

   a) Let E_{ij} be the matrix with a one in the (i, j) position and zeros elsewhere. Show that
   $$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$$
   where δ_{ij} is the Kronecker delta symbol which is one if $i = j$ and zero otherwise.

   b) Using part a) or otherwise find the derived algebra and centre of $gl_n(\mathbb{C})$. You might find it useful to compute things like: $[X, E_{ij}]$, for a general matrix X and $[E_{kk}, E_{ij}]$ and $[E_{ij}, E_{ji}]$.

   c) Show that the upper triangular matrices in $gl_n(\mathbb{C})$ are a Lie subalgebra. Are they an ideal ?

3. Let L be an $\mathbb{F}$ vector space with basis $e_1, \ldots, e_n$.

   a) Let c^i_{jk} be a collection of elements of $\mathbb{F}$ for $i, j, k = 1, \ldots, n$. If $v = \sum_{i=1}^n v^i e_i$ and $w = \sum_{i=1}^n w^i e_i$ define $[v, w] = \sum_{i,j,k=1}^n v^j w^k c^i_{jk} e_i$. Find necessary and sufficient conditions for the c^i_{jk} to satisfy to make this a Lie bracket and hence L a Lie algebra.

   b) If L is a Lie algebra already how do we define the c^i_{jk} so that the bracket of L has the form given in part (a) ?

   c) The quantities c^i_{jk} are called the *structure constants* of L (with respect to the basis $e^1, \ldots, e^n$). Show that two finite dimensional Lie algebras are isomorphic if and only if each has a basis for which it has the same structure constants.

4. Let $\phi: L \to J$ be an isomorphism of Lie algebras. Show that $Z(J) = \phi(Z(L))$ and $J' = \phi(L')$.

5. Denote by su_2 the vector space of all skew-adjoint, traceless, complex 2 by 2 matrices.

   a) Show that su_2 is a real Lie algebra.

   b) Define the Pauli matrices by:
   $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
   and show that $i\sigma_1/2, i\sigma_2/2, i\sigma_3/2$ is a basis of $su(2)$. Compute the structure constants of $su(2)$ for this basis.

   c) Using part a) or by directly defining an isomorphism show that $\mathbb{R}^3$ with the cross product is isomorphic to $su(2)$ as a Lie algebra.

Note: If X is a complex matrix then X is traceless if $\mathrm{tr}(X) = 0$ and X is skew-adjoint if $X^* = -X$ where X^* is the complex conjugate, transpose of X.