**School of Mathematical Sciences**
**PURE MTH 3022**
**Geometry of Surfaces III, Semester 2, 2011**

## Outline of the course

Contents



## 1. Review

We have the usual notation $\mathbb{R}$ for the real numbers, $\mathbb{R}^n$ for $n$-tuples of real numbers and $\mathbb{N} = \{1, 2, 3, \dots\}$ for the natural numbers. Recall that If $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^n)$ are in $\mathbb{R}^n$ then $\|x\| = \sqrt{\sum_{i=1}^n (x^i)^2}$ and $\langle x, y\rangle = \sum_{i=1}^n x^i y^i$. These satisfy

$$\langle x, y\rangle \le \|x\|\|y\| \qquad \textbf{Cauchy's inequality},$$
$$\|x + y\| \le \|x\| + \|y\| \qquad \textbf{Triangle inequality}$$

and

$$\max\{|x^1|, \dots, |x^n|\} \le \|x\| \le \sqrt{n}\max\{|x^1|, \dots, |x^n|\}.$$

*Note* 1.1. Recall from Real Analysis that $\mathbb{R}^n$ with the *Euclidean metric* $d(x,y) = \|x-y\|$ is a metric space. We don't need the metric notion for this course but we will use many of the other notions of metric spaces such as open balls, open sets, sequences, limits and continuous functions but usually only for the metric space $\mathbb{R}^n$. Those of you have who have done Topology and Analysis will recall that $\mathbb{R}^n$ with this norm is an example of a normed vector space.

The open ball around $x \in \mathbb{R}^n$ of radius $\epsilon > 0$ is $B(x,\epsilon) = \{y \in \mathbb{R}^n \mid \|x-y\| < \epsilon\}$ and a subset $U \subset \mathbb{R}^n$ is called *open* if for every $x \in U$ there is some $\epsilon > 0$ such that $B(x,\epsilon) \subset U$.

A *sequence* in $\mathbb{R}^n$ is a function $\mathbb{N} = \{1,2,\dots\} \to \mathbb{R}^n$ usually denoted by its set of values $x_1, x_2, \dots$ or $(x_n)_{n=0}^{\infty}$ or often just $(x_n)$. A sequence $(x_n)$ has *limit* $x \in \mathbb{R}^n$ if for all $\epsilon > 0$ there is an $N$ such that for all $n \geq N$ we have $\|x_n - x\| < \epsilon$. In such a case we also say that $x_n$ converges to $x$ and write $\lim_{n\to\infty} x_n = x$.

**Lemma 1.1.** *A sequence* $(x_n)$ *has* limit $x \in \mathbb{R}^n$ *if and only if* $\lim_{n\to\infty} \|x_n - x\| = 0$.

**Proposition 1.2** (Properties of limits of sequences)**.**

1. *(1) If* $x_m = (x_n^1, \dots, x_m^n) \in \mathbb{R}^n$ *then* $\lim_{m\to\infty} x_m = x = (x^1, \dots, x^n)$ *if and only if* $\lim_{m\to\infty} x_m^i = x^i$ *for all* $i = 1, \dots, n$.
2. *(2) If* $x_n \to x$ *and* $y_n \to y$ *and* $\alpha, \beta \in \mathbb{R}$ *then* $\alpha x_n + \beta y_n \to \alpha x + \beta y$.

**Lemma 1.3.** *If* $x_n \to x$ *then* $\lim_{m,n\to\infty} \|x_n - x_m\| \to 0$.

A sequence with $\lim_{m,n\to\infty} \|x_n - x_m\| \to 0$ is called *Cauchy*.

**Theorem 1.4.** *Every Cauchy sequence in* $\mathbb{R}^n$ *converges.*

**Definition 1.5.** Let $a \in U \subset \mathbb{R}^n$, $U$ open and $f\colon U - \{a\} \to \mathbb{R}^m$. We say that $f$ has limit $L$ at $a$ if $\forall \epsilon > 0\ \exists \delta > 0$ such that if $\|x - a\| < \delta$ and $x \in U$ then $\|f(x) - L\| < \epsilon$.

**Proposition 1.6.** *A function* $f\colon U - \{a\} \to \mathbb{R}^m$ *has limit* $L$ *at* $a$ *if and only if for all sequences* $(x_n) \subset U - \{a\}$ *with* $x_n \to x$ *we have* $f(x_n) \to L$.

**Proposition 1.7** (Properties of limits)**.**

1. *(1) Let* $f\colon U - \{a\} \to \mathbb{R}^m$ *and let* $f(x) = (f^1(x), \dots, f^m(x)$ *where* $f^i\colon U - \{a\} \to \mathbb{R}$ *for each* $i = 1, \dots m$. *Then* $\lim_{x\to a} f(x) = f(a)$ *if and only if for every* $i = 1, \dots m$ *we have* $\lim_{x\to a} f^i(x) = f^i(a)$
2. *(2) Let* $f, g\colon U - \{a\} \to \mathbb{R}^m$ *with* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = J$. *If* $\alpha, \beta \in \mathbb{R}$ *then* $\lim_{x\to a} \alpha f(x) + \beta g(x) = \alpha L + \beta J$.

**Definition 1.8.** Let $U$ be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$. We say that $f$ is *continuous* at $a \in U$ if $\lim_{x\to a} f(x) = f(a)$ and we say that $f$ is *continuous* on $U$ if continuous at every $a \in U$.

**Proposition 1.9.** *A function* $f$ *is continuous at* $a$ *if and only if* $\lim_{x\to a} \|f(x) - f(a)\| = 0$.

**Proposition 1.10.** *A function* $f\colon U \to \mathbb{R}^m$ *is continuous at* $a$ *if and only if for every sequence with* $x_n \to a$ *we have* $f(x_n) \to f(a)$.

**Proposition 1.11** (Properties of continuous functions)**.**

1. *(1) If* $U$ *is open in* $\mathbb{R}^n$ *and* $f, g\colon U \to \mathbb{R}^m$ *are continuous and* $\alpha, \beta \in \mathbb{R}$ *then* $\alpha f + \beta g$ *is continuous.*
2. *(2) If* $U$ *is open in* $\mathbb{R}^n$ *and* $f\colon U \to \mathbb{R}^m$ *then* $f = (f^1, \dots, f^n)$ *is continuous if and only if each* $f^i\colon U \to \mathbb{R}$ *is continuous for every* $i = 1, \dots, n$.
3. *(3) If* $f\colon U \to \mathbb{R}^m$ *and* $g\colon V \to \mathbb{R}^k$ *and* $U$ *is open in* $\mathbb{R}^n$ *and* $V$ *is open in* $\mathbb{R}^k$ *and* $f(U) \subset V$ *then* $f$ *and* $g$ *continuous implies that* $g \circ f$ *is continuous.*

Let $X \subset \mathbb{R}^n$. Recall that $f\colon X \to X$ is called a *contraction* if there exists $0 \leq K < 1$ such that for all $x, y \in X$ we have $\|f(x) - f(y)\| \leq K\|x - y\|$.

**Proposition 1.12** (Contraction mapping theorem)**.** *If* $f\colon \overline{B}(0,r) \to \overline{B}(0,r)$ *is a contraction then there is a unique* $x \in \overline{B}(0,R)$ *such that* $f(x) = x$.

*Note* 1.2. Recall from Real Analysis that the contraction mapping theorem is usually proved for a contraction on a complete metric space. It reduces to this case as $\overline{B}(0,r)$ is a closed subset of the complete metric space $\mathbb{R}^n$ and hence complete.

## 2. Differentiation in $\mathbb{R}^n$.

**Definition 2.1.** Let $U$ be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$. We say that $f$ is differentiable at $a \in U$ if there is a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0.$$

**Lemma 2.2.** *If $f$ is differentiable at $a$ then the $L$ in the definition is unique.*

If $f$ is differentiable at $a$ we denote the linear map $L$ by $f'(a)$. If $f$ is differentiable at every $a \in U$ we say that $f$ is differentiable on $U$.

**Proposition 2.3.** *Let $U$ be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$. Define $f^i\colon U \to \mathbb{R}$ for $i = 1,\dots,m$ by $f(x) = (f^1(x),\dots,f^m(x)$ for all $x \in U$. Then $f$ is differentiable at $a \in U$ if and only if each of the $f^i$ is differentiable at $a$ and $f'(a) = ({f^1}'(a),\dots,{f^m}'(a))$.*

**Lemma 2.4.** *The function $f$ is differentiable at $a$ if and only if there exists a linear function $L$, an $\epsilon > 0$ and a function $R\colon B(0,\epsilon) \to \mathbb{R}^m$ such that $f(a+h) = f(a) + L(h) + R(h)$ and $\lim_{h\to 0} \|R(h)\|/\|h\| = 0$.*

**Proposition 2.5.** *If $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and $v \in \mathbb{R}^m$ then $f(x) = L(x) + v$ is differentiable on all of $\mathbb{R}^n$ and $f'(x) = L$.*

**Proposition 2.6.** *If $f\colon U \to \mathbb{R}^m$ is differentiable at $a$ then $f$ is continuous at $a$.*

**Lemma 2.7.** *If $f$ is differentiable at $a$ then $\forall\epsilon > 0\ \exists\delta > 0$ such that if $\|h\| < \delta$ then $\|f(a+h) - f(h)\| \le (\|f'(a)\| + \epsilon)\|h\|$.*

**Proposition 2.8.** *If $f$ and $g$ are differentiable at $a \in \mathbb{R}^n$ and $\alpha,\beta \in \mathbb{R}$ then $\alpha f + \beta g$ and $fg$ are differentiable at $a$ with $(\alpha f + \beta g)'(a) = \alpha f'(a) + \beta g'(a)$ and $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$.*

**Proposition 2.9** (Chain Rule)**.** *Let $f\colon U \to \mathbb{R}^m$ and $g\colon V \to \mathbb{R}^k$ where $U$ is open in $\mathbb{R}^n$ and $V$ is open in $\mathbb{R}^m$ with $f(U) \subset V$. Then if $f$ is differentiable at $a$ and $g$ is differentiable at $f(a)$ then $g \circ f$ is differentiable at $a$ and $(g \circ f)'(a) = g'(f(a)) \circ f'(a)$.*

**Proposition 2.10.** *If $U \subset \mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$ is differentiable at $a$ and $v \in \mathbb{R}^n$ then*

$$f'(a)(v) = \frac{d}{dt} f(a+tv)\Big|_{t=0}.$$

**Corollary 2.11.** *Let $f$ be as above and write $f = (f^1,\dots,f^m)$. Let $e^i$ be the vector with a $1$ in the $i$th place and zeros elsewhere. Then*

$$f'(a)(e^i) = \left(\frac{\partial f^1}{dx^i}(a),\dots,\frac{\partial f^m}{dx^i}(a)\right).$$

*so that the linear map $f'(a)\colon \mathbb{R}^n \to \mathbb{R}^m$ is the Jacobian matrix*

$$J(f)(a) = \frac{\partial f^i}{\partial x^j}(a).$$

### 2.1. Functions of class $C^k$.

**Definition 2.12.** Let $f\colon U \to \mathbb{R}$ for $U$ open in $\mathbb{R}^n$. We say that $f$ is (of class) $C^k$ if all partial derivatives of $f$ exist and are continuous on $U$ up order $k$. We write $C^0$ for continuous functions and $C^\infty$ or *smooth* for functions which are in $C^k$ for every $k$. The set of all $C^k$ functions on $U$ is denoted by $C^k(U)$ or $C^k(U,\mathbb{R})$.

Let $f\colon U \to \mathbb{R}$ for $U$ open in $\mathbb{R}^n$. We say that $f$ is $C^k$ if each $f^i\colon U \to \mathbb{R}$ is $C^k$ where $f = (f^1,\dots,f^m)$. Again we write $C^k(U,\mathbb{R}^m)$ for the set of all such $f$.

**Proposition 2.13.** *If $f \in C^1(U)$ then $f$ is differentiable at $u$.*

**Proposition 2.14.** *If $f \in C^2(U)$ then*

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$

*on $U$. Similarly if $f$ is $C^k$ for $k \geq 2$ then all partial derivatives up to order including $k$ are independent of order.*

**Proposition 2.15.** *$C^k(U, \mathbb{R}^m)$ is a vector space.*

**Proposition 2.16** (Chain Rule)**.** *Let $f: U \to \mathbb{R}^m$ and $g: V \to \mathbb{R}^k$ where $U$ is open in $\mathbb{R}^n$ and $V$ is open in $\mathbb{R}^m$ with $f(U) \subset V$. Assume that $f$ and $g$ are $C^1$ then*

$$\frac{\partial (g \circ f)^j}{\partial x^i}(a) = \sum_{l=1}^{m} \frac{\partial g^j}{\partial x^l}(f(a)) \frac{\partial f^l}{\partial x^i}(a).$$

*for every $i = 1, \ldots, n$ and $j = 1, \ldots, k$.*

2.2. **Mean Value Theorem.** For any $x_0$ and $x_1$ in $\mathbb{R}^n$ we define $[x_0, x_1]$ to be the line segment joining $x_0$ to $x_1$ that is $\{(1-t)x_0 + tx_1 \mid t \in [0,1]\}$.

**Proposition 2.17** (Mean Value Theorem)**.** *If $U$ is open in $\mathbb{R}^n$ and $f: U \to \mathbb{R}^m$ is differentiable and $[x_0, x_1] \subset U$ and $u \in \mathbb{R}^m$ then $\exists \xi_u \in [x_0, x_1]$ such that*

$$\langle f(x_1), u \rangle = \langle f(x_0), u \rangle + \langle f'(\xi_u)(x_1 - x_0), u \rangle.$$

**Corollary 2.18.** *If $U$ is open in $\mathbb{R}^n$ and $h: U \to \mathbb{R}^m$ is differentiable and $\|h'(\xi)\| \leq \epsilon \ \forall \xi \in [x_0, x_1]$ then $\|h(x_0) - h(x_1)\| < \epsilon \|x_0 - x_1\|$.*

2.3. **Inverse Function Theorem.**

**Theorem 2.19** (Inverse Function Theorem)**.** *Let $U$ be open in $\mathbb{R}^n$ and $f: U \to \mathbb{R}^n$ be $C^k$ for $k \geq 1$. Assume that $f'(a)$ is invertible for some $a \in U$. Then there is an open set $V \subset U$ with $a \in V$ and such that:*
*(1) $f(V)$ is open,*
*(2) $f: V \to f(V)$ is invertible,*
*(3) $f^{-1}$ is $C^k$, and*
*(4) $(f^{-1})'(f(a)) = [f'(a)]^{-1}$.*

**Corollary 2.20** (Open mapping theorem)**.** *Let $U$ be open in $\mathbb{R}^n$ and $f: U \to \mathbb{R}^n$ be such that $f'(x)$ is invertible for all $x \in U$. Then $f(U)$ is open in $\mathbb{R}^n$.*

**Definition 2.21.** If $U$ and $V$ are open in $\mathbb{R}^n$ and $f: U \to V$ is $C^k$ with a $C^k$ inverse then $f$ is called a *($C^k$) diffeomorphism*.

For the implicit function theorem we need the following notation. If $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ then we denote by $(x, y)$ the obvious element $\mathbb{R}^n \times \mathbb{R}^m$.

**Theorem 2.22** (Implicit Function Theorem)**.** *Let $U$ be open in $\mathbb{R}^{n+m}$, $(x_0, y_0) \in U$ and $F: U \to \mathbb{R}^m$ be $C^k$. If $F(x_0, y_0) = 0$ and*

$$\frac{\partial F}{\partial y}(x_0, y_0) = \begin{pmatrix} \frac{\partial F^1}{\partial y^1}(x_0, y_0) & \ldots & \frac{\partial F^1}{\partial y^m}(x_0, y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial y^1}(x_0, y_0) & \ldots & \frac{\partial F^m}{\partial y^m}(x_0, y_0) \end{pmatrix}$$

*is non-singular then there exists an open set $V \subset \mathbb{R}^n$ containing $x_0$ and a $C^k$ function $f: V \to \mathbb{R}^m$, such that $F(x, f(x)) = 0$ for all $x \in V$.*

## 3. Submanifolds

**Definition 3.1.** A subset $S \subset \mathbb{R}^N$ is called a *submanifold* of dimension $n$ if for all $s \in S$ there exists a $U$ open in $\mathbb{R}^N$, containing $s$, and a smooth map $\phi: U \to \mathbb{R}^n$ such that $\phi(U)$ is open, $\phi: U \to \phi(U)$ is a diffeomorphism and

$$S \cap U = \{x \in U \mid (\phi^{n+1}(x), \ldots, \phi^N(x)) = 0\}.$$

**Theorem 3.2.** *Let $S \subset \mathbb{R}^N$ then the following are equivalent.*
*(1) $S$ is a submanifold of dimension $n$;*
*(2) for every $s \in S$ there is an open set $U \subset \mathbb{R}^N$ containing $s$ and a smooth function $F\colon U \to \mathbb{R}^{N-n}$ such that $F'(x)$ is onto for all $x \in S \cap U$ and $S \cap U = \{x \in U \mid F(x) = 0\}$;*
*(3) for all $s \in S$ there exists $U$ open in $\mathbb{R}^N$ such that $S \cap U$ is the graph of a smooth function of $n$ of the $N$ variables;*
*(4) for all $s \in S$ there is a $V$ open in $\mathbb{R}^N$ containing $s$ and $U$ open in $\mathbb{R}^n$ and a $\psi\colon U \to V$ which is one to one with $\psi'(x)$ one to one for all $x \in U$ and such that $S \cap V = \psi(U)$.*

**Corollary 3.3.** *If $U$ is open in $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^{N-n}$ is smooth with $F'(x)$ onto for all $x \in S = F^{-1}\{0\}$ then $S$ is a submanifold of dimension $n$.*

**Definition 3.4.** Let $S \subset \mathbb{R}^N$ be an $n$ dimensional submanifold. If $U$ is an open set in $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^{N-n}$ is smooth with $F'(s)$ onto for all $s \in S \cap U$ and $S \cap U = \{s \in U \mid F(s) = 0\}$ then $F$ is called a *local defining equation* for $S$.

**Definition 3.5.** Let $S \subset \mathbb{R}^N$ be an $n$ dimensional submanifold. If $U$ is an open subset of $\mathbb{R}^n$ and $V$ an open subset of $\mathbb{R}^N$ and $\psi\colon U \to V$ is smooth and one to one with $\psi'(x)$ one to one for all $x \in U$ and $\psi(U) = S \cap V$ then $\psi$ is called a *local parametrisation* of $S$.

### 3.1. Tangent space to a submanifold.

**Definition 3.6.** Let $s$ be a point in a submanifold $S \subset \mathbb{R}^N$. Let $\epsilon > 0$. Then a smooth map $\gamma\colon (-\epsilon, \epsilon) \to S \subset \mathbb{R}^N$ with $\gamma(0) = s$ is a called a smooth path in $S$ through $s$.

**Definition 3.7.** Define $T_sS$ to be the union of all the vectors $\gamma'(0)$ for $\gamma$ a smooth path in $S$ through $s$. Call it the tangent space to $S$ at $s$.

**Proposition 3.8.** *$T_sS$ is an $n$-dimensional subspace of $\mathbb{R}^N$.*

**Proposition 3.9.** *If $F$ is a local defining equation for $S$ defined on an open set containing $s$ then $T_sS = \ker F'(s)$. If $\psi$ is a local parametrisation for $S$ with $\psi(x) = s$ then $T_sS = im\, \psi'(x)$. Moreover $\frac{\partial \psi}{\partial x^1}(x), \dots, \frac{\partial \psi}{\partial x^n}(x)$ are a basis for $T_sS$.*

### 3.2. Smooth functions on submanifolds.

Let $S \subset \mathbb{R}^N$ be a submanifold and let $f\colon S \to \mathbb{R}$ be a function.

**Definition 3.10.** We say that $f$ is a smooth function if there is an open set $U \subset \mathbb{R}^N$ with $S \subset U$ and a smooth function $\hat{f}\colon U \to \mathbb{R}$ such that for any $s \in S$ we have $\hat{f}(s) = f(s)$.

**Proposition 3.11.** *Let $S \subset \mathbb{R}^N$ be a submanifold and $f\colon S \to \mathbb{R}$ a function.*
*(1) If $\psi\colon U \to S$ is a parametrisation and $f$ is smooth then $f \circ \psi\colon U \to \mathbb{R}$ is smooth.*
*(2) If for every $s \in S$ there is a parametrisation $\psi\colon U \to S$ with $s \in \psi(U)$ such that $f \circ \psi\colon U \to \mathbb{R}$ is smooth then $f$ is smooth.*

**Proposition 3.12.** *Let $S \subset \mathbb{R}^N$ be a submanifold and $\psi\colon U \to S$ and $\chi\colon V \to S$ be parametrisations with $\psi(U) = \chi(V)$ then $\psi^{-1} \circ \chi\colon V \to U$ is a diffeomorphism.*

Let $S$ be a submanifold and $f\colon S \to \mathbb{R}$ a smooth map. Let $\hat{f}\colon U \to \mathbb{R}$ be a smooth extension of $f$ to an open set $U$ containing $S$. Then $\hat{f}'(s)\colon \mathbb{R}^N \to \mathbb{R}$ and we denote by $f'(s)\colon T_sS \to \mathbb{R}$ the restriction of $\hat{f}'(s)$ to $T_sS \subset \mathbb{R}^N$.

**Lemma 3.13.** *$f'(s)\colon T_sS \to \mathbb{R}$ is independent of the choice of extension $\hat{f}$.*

**Definition 3.14.** If $f\colon S \to \mathbb{R}^m$ and each $f^i\colon S \to \mathbb{R}$ is smooth where $f = (f^1, \dots, f^m)$ then we say that $f$ is smooth.

**Proposition 3.15.** *If $f\colon S \to \mathbb{R}^m$ is a smooth map from a submanifold $S$ which has its image inside a submanifold $T \subset \mathbb{R}^m$ then for any $s \in S$ we have that $f'(s)(T_sS) \subset T_{f(s)}T$.*

## 4. Geometry of curves

**Definition 4.1.** A curve is a one-dimensional submanifold.

If $c$ is a point in a curve $C$ then $T_cC$ is one-dimensional so that $T_cC - \{0\}$ has two connected components. A continuous choice of one of these two components at each point of $C$ is called an *orientation* and a curve with an orientation is called an oriented curve.

**Definition 4.2.** A parametrised curve $C$ is a curve for which there is a parametrisation $\gamma\colon (a,b) \to C$ with $\gamma(a,b) = C$.

For a parametrisation $\gamma'(t) \neq 0$. If $\gamma'(t)$ is in the chosen half of $T_{\gamma(t)}C$ for an oriented curve $C$ then we say the parametrisation is oriented.

**Definition 4.3.** We say a parametrised curve is parametrised by arc length if $\|\gamma(t)\| = 1$ for all $t$.

**Lemma 4.4.** *If $\gamma(t)$ and $\tilde{\gamma}(t)$ are two arc length parametrisations of a curve $C$ then there is a $t_0 \in \mathbb{R}$ such that $\gamma(t) = \tilde{\gamma}(t + t_0)$ for all $t$.*

**Definition 4.5.** Then unit normal vector field $T$ on a parametrised curve $C$ is defined by $T = \gamma'(t)/\|\gamma'(t)\|$.

**Lemma 4.6.** *If $\gamma(t)$ is parametrised by arc length then $\langle \gamma''(t), \gamma'(t)\rangle = 0$.*

**Definition 4.7.** If $C$ is a curve with an arc length parametrisation $\gamma(t)$ then the curvature of $C$ at $c = \gamma(t)$ is $\kappa(c) = \|\gamma''(t)\|$.

**Proposition 4.8.** *If $C$ is a curve and $\gamma(t)$ is a (not necessarily arc-length) parametrisation then*

$$\begin{aligned}\kappa &= \frac{1}{\|\gamma'\|^2}\left\|\gamma'' - \gamma'\frac{\langle \gamma', \gamma''\rangle}{\|\gamma'\|^2}\right\| \\ &= \frac{1}{\|\gamma'\|^2}\left(\|\gamma''\|^2 - \frac{\langle \gamma', \gamma''\rangle^2}{\|\gamma'\|^2}\right)^{1/2}\end{aligned}$$

### 4.1. **Curves in $\mathbb{R}^3$.**

**Definition 4.9.** Let $C \subset \mathbb{R}^3$ be a curve. Define $N = T'/\|T'\|$ the *principal normal* to $\gamma$ and $B = T \times N$ the *unit binormal.*

**Proposition 4.10** (Frenet formula)**.** *Let $\dot{T}$, $\dot{N}$ and $\dot{B}$ denote differentiation with respect to arc-length. Then we have*

$$\dot{T} = \kappa N \qquad \dot{N} = -\kappa T + \tau B \qquad \dot{B} = -\tau N$$

*for a function $\tau$ on the curve called the* torsion *of the curve.*

## 5. Geometry of surfaces

Let $\Sigma \subset \mathbb{R}^3$ be a surface. Then $T_s\Sigma^\perp$, the orthogonal space to the tangent space $T_s\Sigma$, is one-dimensional so if zero is removed there are two connected halves. In other words there are two possible unit normals. An orientation for $\Sigma$ is a choice of unit normal $n(s) \in T_s\Sigma^\perp$ continuously across the surface. An oriented surface is a surface with an orientation. If $\psi\colon U \to \Sigma$ is a parametrisation we say it is oriented if

$$n = \frac{\frac{\partial\psi}{\partial x^1} \times \frac{\partial\psi}{\partial x^2}}{\left\|\frac{\partial\psi}{\partial x^1} \times \frac{\partial\psi}{\partial x^2}\right\|}.$$

The unit normal defines a map $n\colon \Sigma \to S^2$ called the *Gauss map.*

**Proposition 5.1.** *Let $s \in \Sigma$ and let $v \in T_s\Sigma$. Let the unit normal at $s$ be $n$. Then $\exists \epsilon > 0$ and a map $\gamma\colon (-\epsilon, \epsilon) \to \Sigma$ such that $\gamma(0) = s$, $\gamma'(0) = v$ and $\gamma(t) = a(t)v + b(t)n$ for some functions $a$ and $b$.*

**Proposition 5.2.** *The curvature of the curve $\gamma$ in Prop. 5.1 is given by $\langle n, \gamma''(0)\rangle/\|\gamma''(0)\|^2$.*

If $\psi\colon U \to \Sigma$ are local parameters for a surface $\Sigma$ with $\psi(x) = s$ and $v$ and $w$ are in $T_s\Sigma$ define the *second fundamental form* $\alpha$ by

$$\alpha(s)(v,w) = \sum_{i,j=1}^{2} v_i w_j \left\langle \frac{\partial^2\psi}{\partial x^i \partial x^j}(x), n\right\rangle$$

where $v = \sum_{i=1}^2 v_i \frac{\partial\psi}{\partial x^i}$ and $w = \sum_{i=1}^2 w_i \frac{\partial\psi}{\partial x^i}$.

**Proposition 5.3.** *Let $\alpha$ be the second fundamental form at a point $s$ of a surface $\Sigma$ then:*
*(1) $\alpha(v, w) = -\langle dn(v), w)\rangle = -\langle dn(w), v\rangle$,*
*(2) $\alpha$ is independent of the parametrisation,*
*(3) if $\|v\| = 1$ and $\gamma$ is chosen as in Prop. 5.1 then $\alpha(v, v)$ is the curvature of $\gamma$ at $s$.*

**Definition 5.4.** The first fundamental form at $s \in \Sigma$ is the inner product $g(v, w) = \langle v, w\rangle$.

**Definition 5.5.** Define a function $\Pi(s)\colon T_s\Sigma \to T_s\Sigma$ by $\Pi = -dn(s) = -n'(s)$.

Clearly $\alpha(v, w) = \langle \Pi(v), w)\rangle$. $\Pi$ is symmetric so it has orthogonal eigenvectors $v_1$ and $v_2$ with eigenvalues $\lambda_1$ and $\lambda_2$. The directions $v_i$ are the directions of greatest and least curvature. The eigenvalues $\lambda_1$, $\lambda_2$ are called the *principal curvatures*. Their average $(1/2)\operatorname{tr}(\Pi)$ is called the *mean curvature* and their product $\det(\Pi)$ is called the Gaussian curvature.

**Proposition 5.6.** *If $v_1$ and $v_2$ are a basis for $T_s\Sigma$ and $\alpha_{ij} = \alpha(v_i, v_j)$ and $g_{ij} = g(v_i, v_j) = \langle v_i, v_j\rangle$ then $\Pi = \alpha g^{-1}$ so that $\det(\Pi) = \det(\alpha_{ij})/\det(g_{ij})$.*

## 6. Integration

6.1. **Integration in $\mathbb{R}^n$.** Let $R$ be a closed bounded subset of $\mathbb{R}^2$ and $f\colon R \to \mathbb{R}$ a continuous function. Define a *partition* $\mathcal{P}$ to be a collection of rectangles $R_i$, $i = 1, \dots, n$ which intersect at most on their edges and which cover $R$, that is $R \subset \bigcup_{i=1}^n R_i$. Denote the area of each $R_i$ by $\triangle A_i$. Define a *selection* for $\mathcal{P}$ to be a collection of points $x_i^* \in R_i \cap R$ for each $i$. For each partition and selection consider the sum

$$\sum_{i=1}^N f(x_i^*)\triangle A_i.$$

Let $\operatorname{mesh}(\mathcal{P}) = \min\{\triangle A_i\}$.

**Theorem 6.1.** *There exists a number $L$ such that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\operatorname{mesh}(\mathcal{P}) < \delta$ and $x_i^*$ is any selection then $\left|\sum_{i=1}^N f(x_i^*)\triangle A_i - L\right| < \epsilon$.*

The number $L$ is denoted $\int_R f(x)dA$ and called the integral of $f$ over $R$.

**Proposition 6.2.** *The integral satisfies:*
*(i) $\int_R f dA$ is linear in $f$,*
*(ii) if $f(x, y) \le g(x, y)$ for all $(x, y) \in R$ then $\int_R f(x, y)dA \le \int_R g(x, y)dA$ with equality if and only $f(x, y) = g(x, y)$ for all $(x, y) \in R$,*
*(iii) if $f(x, y) \ge 0$ then $\int_R f(x, y)dA$ is the volume in $\mathbb{R}^3$ of the region consisting of all $(x, y, z)$ such that $(x, y) \in R$ and $0 \le z \le f(x, y)$ and*
*(iv) if $R_1$ and $R_2$ are two regions with $R_1 \cap R_2 = \emptyset$ and $R = R_1 \cup R_2$ then $\int_{R_1} f dA + \int_{R_2} f dA = \int_R f dA$.*

**Theorem 6.3** (Fubini)**.** *Let $R$ be a closed bounded region in $\mathbb{R}^2$ and $f\colon R \to \mathbb{R}$ a continuous function. Then*

$$\int\left(\int f(x, y)dx\right)dy = \int_R f dA = \int\left(\int f(x, y)dy\right)dx.$$

All these results generalise to $\mathbb{R}^n$.

**Definition 6.4.** Let $U$ be an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ be continuous and define the *support* of $f$ ($\operatorname{supp}(f)$) to be the closure in $U$ of $\{x \in U\colon f(x) \neq 0\}$.

**Theorem 6.5** (Change of variable formula)**.** *Let $U$ and $V$ be open subsets of $\mathbb{R}^n$ and let $\psi\colon U \to V$ be a diffeomorphism. Let $f\colon V \to \mathbb{R}$ have support which is closed and bounded in $\mathbb{R}^n$ then*

$$\int_U f dx^1 \dots dx^n = \int_V f \circ \psi |\det(J(\psi))| dx^1 \dots dx^n$$

*where*

$$J(\psi) = \begin{pmatrix} \frac{\partial \psi^1}{\partial x^1} & \cdots & \frac{\partial \psi^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi^n}{\partial x^1} & \cdots & \frac{\partial \psi^n}{\partial x^n} \end{pmatrix}$$

*is the Jacobian matrix of $\psi$.*

6.2. **Volume forms and integration.** Let $V$ be a vector space of dimension $n$. An $n$-form is a multilinear and totally antisymmetric map

$$\omega\colon \underbrace{V \times \cdots \times V}_{\text{n times}} \to \mathbb{R}.$$

Multilinear means linear in each of the $n$ factors separately, that is for any $w_i, v_1, \dots, v_n \in V$ and $a, b \in \mathbb{R}$ we have

$$\omega(v_1, \dots, av_i + bw_i, \dots, v_n) = a\omega(v_1, \dots, v_i, \dots, v_n) + b\omega(v_1, \dots, w_i, \dots, v_n).$$

Totally antisymmetric means that if $\pi$ is a permutation of the numbers $1, \dots, n$ with sign[1] denoted by $\operatorname{sign}(\pi)$ then

$$\omega(v_{\pi(1)}, \dots, v_{\pi(n)}) = \operatorname{sign}(\pi)\omega(v_1, \dots, v_n).$$

Notice that if $\omega$ is an $n$-form then

$$\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

and hence

$$\omega(v_1, \dots, v, \dots, v, \dots, v_n) = 0.$$

If $v_1, \dots, v_n$ is a basis of $V$ we define an $n$-form $[v_1, \dots, v_n]$ by

$$[v_1, \dots, v_n](w_1, \dots, w_n) = \det(X_{ij})$$

where $w_i = \sum_j^n X_{ij} v_j$ for each $i = 1, \dots, n$.

We denote the set of all $n$-forms by $\det(V^*)$. It is a vector space and we have

**Proposition 6.6.** *The space of all $n$-forms,* $\det(V^*)$*, is one dimensional. If $\omega$ is an $n$-form and $v_1, \dots, v_n$ is a basis then $\omega = \omega(v_1, \dots, v_n)[v_1, \dots, v_n]$.*

**Corollary 6.7.** *Let $w_1, \dots, w_n$ and $v_1, \dots, v_n$ be bases of $V$. Let $w_i = \sum_{j=1}^n X_{ij} v_j$, then*

$$[v_1, \dots, v_n] = \det(X_{ij})[w_1, \dots, w_n].$$

If $\Sigma \subset \mathbb{R}^N$ is an $n$ dimensional submanifold then each tangent space $T_s\Sigma$ is $n$ dimensional and we can form $\det(T_s\Sigma^*)$ the space of all $n$-forms on $T_s\Sigma$. As $\det(T_s\Sigma^*)$ is one dimensional it follows that $\det(T_s\Sigma^*) - \{0\}$ has two connected components. We call $\Sigma$ oriented if we have picked one of these two components at each $s$ on $\Sigma$ in a continuous manner. If $\omega$ is in the chosen half of $\det(T_s\Sigma^*) - \{0\}$ then we call it positive. If $\psi\colon U \to \Sigma$ is a parametrisation then we say it is oriented if the $n$ form $[\frac{\partial\psi}{\partial x^1}, \dots, \frac{\partial\psi}{\partial x^n}]$ is positive.

An $n$ form $\omega$ on an $n$ dimensional submanifold is a smooth choice of an $\omega(s) \in \det(T_s\Sigma^*)$ for every $s \in \Sigma$. To say what smooth means we choose a parametrisation $\psi\colon U \to \Sigma$. Then if $\psi(x) = s$ we have

$$\omega(s) = \omega_\psi(s)\left[\frac{\partial\psi}{\partial x^1}(x), \dots, \frac{\partial\psi}{\partial x^n}(x)\right]$$

where from Prop. 6.6 we have that

$$\omega_\psi(s) = \omega(s)\left(\frac{\partial\psi}{\partial x^1}(\psi^{-1}(s)), \dots, \frac{\partial\psi}{\partial x^n}(\psi^{-1}(s))\right).$$

We call $\omega$ smooth if whenever we choose a parametrisation like this the function $\omega_\psi\colon \psi(U) \to \mathbb{R}$ is smooth.

**Proposition 6.8.** *If $\chi\colon V \to \Sigma$ and $\psi\colon U \to \Sigma$ are parametrisations and $\chi(V) = \psi(U)$ then*

$$\omega_\psi = \left(\det(J(\chi^{-1} \circ \psi)) \circ \psi^{-1}\right)\omega_\chi.$$

*If $\chi$ and $\psi$ are both oriented parametrisations then* $\det(J(\chi^{-1} \circ \psi)) > 0$.

*Note* 6.1. This proposition shows that if $\omega_\chi$ is smooth then so also is $\omega_\psi$.

**Definition 6.9.** If $\omega$ is a smooth $n$-form on a submanifold $\Sigma$ we define its support to be the closure of the set of points at which it is not zero.

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[1]If $\hat\pi$ is the matrix whose $(i, j)$th entry is 1 if $\pi(i) = j$ and zero otherwise then $\operatorname{sign}(\pi) = \det(\hat\pi)$.

**Definition 6.10.** If $\psi\colon U \to \Sigma$ is an oriented parametrisation and $\omega$ a smooth $n$ form with support in $\psi(U)$ we define

$$I_\psi(\omega) = \int_U \omega_\psi \circ \psi dx^1 \dots dx^n.$$

**Proposition 6.11.** *If $\chi\colon V \to \Sigma$ is another oriented parametrisation then $I_\psi(\omega) = I_\chi(\omega)$.*

*Note* 6.2. To be sure that the integral exists we should really require that the support of $\omega$ be compact, ie closed and bounded in $\mathbb{R}^N$.

**Definition 6.12.** Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of $\Sigma$. A partition of unity subordinate to this cover is a collection of smooth functions $\rho_\alpha\colon \Sigma \to [0,\infty) \subset \mathbb{R}$ such that $\operatorname{supp}(\rho_\alpha) \subset U_\alpha$ and $\sum_\alpha \rho_\alpha = 1$.

*Note* 6.3. To make sure the sum in the integral makes sense we should impose a condition called local finiteness. For this purposes of this course the sums can be assumed to be finite so we will ignore this point.

If $\omega$ is an $n$ form with support in the image of some (oriented) parametrisation $\psi\colon U \to \Sigma$ then we define

$$\int_\Sigma \omega = I_\psi(\omega).$$

If $\omega$ is a more general form we assume that it is possible to find a collection of parametrisations $\psi_\alpha\colon U_\alpha \to \Sigma$ with $\Sigma = \bigcup \psi_\alpha(U_\alpha)$ and a partition of unity subordinate to $\{\psi_\alpha(U_\alpha)\}$ and we define

$$\int_\Sigma \omega = \sum_\alpha \int_M \rho_\alpha \omega.$$

*Note* 6.4. (1) The support of $\rho_\alpha\omega$ is in $\psi_\alpha(U_\alpha)$.
(2) The existence of the partition of unity required is guaranteed in much generality which we will not go into in this course. We shall assume it exists.

**Proposition 6.13.** *The integral of an $n$ form just defined is independent of the choice of parametrisations and partition of unity.*

6.3. **Volume forms.**

**Lemma 6.14.** *If $V$ is a vector space with an inner product $\langle\ ,\ \rangle$ and $e_1,\dots,e_n$ and $f_1,\dots,f_n$ are orthonormal bases then $[e_1,\dots,e_n] = \pm[f_1,\dots,f_n]$.*

**Definition 6.15.** If $\Sigma \subset \mathbb{R}^N$ is an oriented submanifold and $e_1,\dots,e_n$ is an orthonormal basis of $T_s\Sigma$ and $[e_1,\dots,e_n]$ is positive we define $\mathrm{vol}_\Sigma(s) = [e_1,\dots,e_n]$, the volume form of $\Sigma$.

*Note* 6.5. (1) Orthonormal means orthonormal with respect to the inner product on $\mathbb{R}^N$ restricted to $T_s\Sigma$.
(2) The Lemma guarantees that the volume is independent of the choice or orthonormal basis.
(3) When its obvious from context we will drop the $\Sigma$ from $\mathrm{vol}_\Sigma$ and just write vol.

**Proposition 6.16.** *Let $\psi\colon U \to \Sigma$ be a parametrisation of $\Sigma$. Let $g_{ij} = \langle \frac{\partial\psi}{\partial x^i}, \frac{\partial\psi}{\partial x^j}\rangle$ then*

$$\mathrm{vol}_\Sigma = \det(g_{ij})^{-1/2}\left[\frac{\partial\psi}{\partial x^1},\dots,\frac{\partial\psi}{\partial x^n}\right].$$

*Note* 6.6. It follows that $\mathrm{vol}_\Sigma$ is a smooth $n$ form.

**Lemma 6.17.** *If $V \subset \mathbb{R}^3$ is a two dimensional subspace and $\omega$ is any two form on $V$ then there is an $n \in V^\perp$ (not necessarily unit length) such that for any $v$ and $w$ in $V$ we have $\omega(v,w) = \langle v \times w, n\rangle$.*

*Note* 6.7. This lemma shows that orienting a surface by chosing a normal is the same as orienting a surface by choosing one half of $\det(T_s\Sigma^*)$.

**Proposition 6.18.** *If $\Sigma \subset \mathbb{R}^3$ is an oriented surface and $n$ is the unit normal to $\Sigma$ at $s$ which defines the orientation then*

$$\mathrm{vol}_\Sigma(v,w) = \langle v \times w, n\rangle$$

*for any $v$ and $w$ in $T_s\Sigma$.*

6.4. **One forms.** Let $\Sigma$ be a submanifold and $f\colon \Sigma \to \mathbb{R}$ a smooth function. Then for any $s \in \Sigma$ we have that $df(s) = f'(s)\colon T_s\Sigma \to \mathbb{R}$ is a linear map. In particular if $\psi\colon U \to \Sigma$ is a parametrisation and $\hat{\psi} = \psi^{-1}$ then each of the components of $\hat{\psi} = (\hat{\psi}^1, \dots, \hat{\psi}^n)$ is a function $\hat{\psi}^i\colon \psi(U) \to \mathbb{R}$ and hence $d\hat{\psi}^i(s)\colon T_s\Sigma \to \mathbb{R}$ is a linear map for any $s \in \psi(U)$. We have

**Proposition 6.19.** *The linear maps $d\hat{\psi}^1(s), \dots, d\hat{\psi}^n(s)$ are a basis of the dual space $T_s\Sigma^*$ satisfying*

$$d\hat{\psi}^i(s)\left(\frac{\partial \psi}{\partial x^j}(x)\right) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

*if $\psi(x) = s$.*

**Definition 6.20.** A one form $\eta$ is a choice of linear map $\eta(s)\colon T_s\Sigma \to \mathbb{R}$ for every $s$. Any one form can be expanded as $\eta(s) = \sum_{i=1}^n \eta_i(s) d\hat{\psi}^i(s)$ and we say $\eta$ is smooth if each of the $\eta_i\colon \psi(U) \to \mathbb{R}$ is smooth.

**Proposition 6.21.** *If $\psi\colon U \to \Sigma$ and $\chi\colon V \to \Sigma$ are parametrisations and $\psi(U) = \chi(V)$ then*

$$d\hat{\chi}^i(s) = \sum_{j=1}^n \frac{\partial(\chi^{-1} \circ \psi)^i}{\partial x^j}(\psi^{-1}(s)) d\hat{\psi}^j(s)$$

*so that if $\eta(s) = \sum_{i=1}^n \eta_i(s) d\hat{\psi}^i(s)$ and $\tilde{\eta}(s) = \sum_{i=1}^n \tilde{\eta}_i(s) d\hat{\chi}^i(s)$ then*

$$\eta_i(s) = \sum_{j=1}^n \tilde{\eta}_j(s) \frac{\partial(\chi^{-1} \circ \psi)^j}{\partial x^i}(\psi^{-1}(s)).$$

**Definition 6.22.** If $V$ is a two dimensional vector space and $\alpha$ and $\beta$ are one forms we define $\alpha \wedge \beta$, their *wedge product*, by

$$(\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v).$$

for any $v$ and $w$ in $V$.

**Lemma 6.23.** *If $\psi\colon U \to \Sigma$ is a parametrisation of a surface $\Sigma$ then*

$$d\hat{\psi}^1 \wedge d\hat{\psi}^2 = \left[\frac{\partial \psi}{\partial x^1}, \frac{\partial \psi}{\partial x^2}\right].$$

**Proposition 6.24.** *(1) If $\psi\colon U \to \Sigma$ is a parametrisation of a surface $\Sigma$ and $\eta$ is a one form then the two form $d\eta$ defined by*

$$\begin{aligned} d\eta(s) &= \sum_{i=1}^2 d\eta_i(s) \wedge d\hat{\psi}^i(s) \\ &= \left(\frac{\partial \eta_2 \circ \psi}{\partial x^1}(\psi^{-1}(x)) - \frac{\partial \eta_1 \circ \psi}{\partial x^2}(\psi^{-1}(x))\right) d\hat{\psi}^1(s) \wedge d\hat{\psi}^2(s). \end{aligned}$$

*is independent of the parametrisation.*
*(2) If $f$ is a function on $\Sigma$ then $d(f\eta) = df \wedge \eta + f d\eta$.*

**Definition 6.25.** A closed surface is a two dimensional submanifold of $\mathbb{R}^3$ which is closed and bounded. It follows that it has no boundary.

**Proposition 6.26** (Weak Green's Theorem). *If $\eta$ is a one form on an oriented closed surface $\Sigma$ then $\int_\Sigma d\eta = 0$.*

## 7. Gauss-Bonnet Theorem

**Definition 7.1** (Pulling back two forms). If $f\colon \Sigma_1 \to \Sigma_2$ is a smooth map between two submanifolds of dimension $n$ and $\omega$ is an $n$ form on $\Sigma_2$ we define an $n$ form on $\Sigma_1$ called $f^*(\omega)$, the pull-back of $\omega$, as follows. If $v_1, \dots, v_n$ are in $T_s\Sigma$ then $f'(s)(v_1), \dots, f'(s)(v_n)$ are in $T_{f(s)}\Sigma_2$ and so we define

$$f^*(\omega)(s)(v_1, \dots, v_n) = \omega(f(s))(f'(s)(v_1), \dots, f'(s)(v_n)).$$

If $\psi\colon U \to \Sigma_1$ is a local parametrisation and $v = \sum_{i=1}^n v^i \frac{\partial \psi}{\partial x^i}$ and $w = \sum_{i=1}^n w^i \frac{\partial \psi}{\partial x^i}$ then $f'(s)(v) = \sum_{i=1}^n v^i \frac{\partial f \circ \psi}{\partial x^i}$ and $f'(s)(w) = \sum_{j=1}^n w^j \frac{\partial f \circ \psi}{\partial x^j}$ and hence

$$f^*(\omega)(v, w) = \sum_{i,j=1}^n v^i w^j \omega(f(s))\left(\frac{\partial f \circ \psi}{\partial x^i}, \frac{\partial f \circ \psi}{\partial x^j}\right).$$

**Proposition 7.2.** *If $\Sigma \subset \mathbb{R}^3$ is a surface with Gaussian curvature $R$ and $n\colon \Sigma \to S^2$ is the unit normal then*

$$R \operatorname{vol}_\Sigma = n^*(\operatorname{vol}^2_S).$$

*Note* 7.1. We choose orientations here so that $\operatorname{vol}_\Sigma(v, w) = \langle v \times w, n\rangle$ and similarly for $S^2$ with $n$ there being the outward normal.

**Proposition 7.3.** *Let $\Sigma_t$ be a family of closed oriented surfaces in $\mathbb{R}^3$ depending smoothly on a parameter $t$. Define $\eta(X) = \langle \frac{dn}{dt} \times dn(X), n\rangle$ then*

$$\frac{d}{dt} R_t \operatorname{vol}_t = d\eta.$$

**Corollary 7.4.** *Let $\Sigma_t$ be a family of closed oriented surfaces in $\mathbb{R}^3$ depending smoothly on a parameter $t$. Then $\int_{\Sigma_t} R_t \operatorname{vol}$ is independent of $t$.*

**Proposition 7.5.** *For a sphere we have*

$$\frac{1}{2\pi}\int_{S^2} R \operatorname{vol} = 2.$$

**Definition 7.6.** We say a surface $\Sigma'$ is obtained from a surface $\Sigma$ by *adding a handle* if we remove two disks from $\Sigma$ and attach to the two resulting circles in $\Sigma$ each end of a cylinder.

**Proposition 7.7.** *If the oriented closed surface $\Sigma'$ is obtained from the oriented closed surface $\Sigma$ by adding a handle then*

$$\frac{1}{2\pi}\int_{\Sigma'} R \operatorname{vol} = \frac{1}{2\pi}\int_{\Sigma} R \operatorname{vol} - 2$$

**Corollary 7.8.** *If $\Sigma$ is obtained from a sphere by adding $g$ handles then*

$$\frac{1}{2\pi}\int_{\Sigma} R \operatorname{vol} = 2 - 2g.$$

**Theorem 7.9.** *If $\Sigma$ is a closed surface in $\mathbb{R}^3$ then $\Sigma$ is homeomorphic to a sphere with $g$ handles.*

*Note* 7.2. The quantity $g = g(\Sigma)$ is called the genus of $\Sigma$ and can be calculated from

$$g(\Sigma) = \frac{1}{2}\left(2 - \frac{1}{2\pi}\int_{\Sigma} R \operatorname{vol}\right)$$

### 7.1. **Tessellations.**

**Definition 7.10.** Let $\Sigma$ be a surface in $\mathbb{R}^3$. A tessellation $T$ for $\Sigma$ is:
(1) a set of points $v_1, \dots, v_n$ in $\Sigma$ called vertices,
(2) a collection of curves $e_1, \dots, e_m$ called edges which join vertices such that (i) every vertex has at least two edges ending at it and (ii) edges meet only at vertices, and
(3) a collection of subsets $f_1, \dots, f_r$ of $\Sigma$ called faces such that (i) the boundary of every face is a union of edges and each face is homeomorphic to a polygon in $\mathbb{R}^2$.

A tessellation where every face has three edges is called a triangulation.

**Definition 7.11.** Let $T$ be a tessellation of a surface $\Sigma$ and let $v$ be the number of vertices, $e$ the number of edges and $f$ the number of faces. Define

$$\chi(\Sigma, T) = v - e + f.$$

**Proposition 7.12.** *Let $\Sigma$ be a surface and $T$ and $T'$ tessellations. Then $\chi(\Sigma, T) = \chi(\Sigma, T')$.*

**Definition 7.13.** If $\Sigma$ is a surface we define its Euler characteristic $\chi(\Sigma)$ to be $\chi(\Sigma, T)$ for some some tessellation.

**Proposition 7.14.** *If $\Sigma$ is a surface of genus $g$, ie it is obtained from a sphere by adding $g$ handles then $\chi(\Sigma) = 2 - 2g$.*

*Note* 7.3. This gives an alternative way of calculating $g(\Sigma)$ namely $g(\Sigma) = (1/2)(2 - \chi(\Sigma)$.

**Theorem 7.15** (Gauss-Bonnet). *If $\Sigma$ is a closed oriented surface then*

$$\frac{1}{2\pi}\int_{\Sigma} R \operatorname{vol} = \chi(\Sigma) = 2 - 2g(\Sigma).$$