Solutions to Mock Exam 2011 — Geometry of Surfaces ①

1(a) F. $\gamma'(t) = (1, 2t, 99t^{98})$

$\|\gamma'(t)\|^2 = 1 + 4t^2 + (99t^{98})^2 > 1$

$\therefore$ not arc-length parametrised. F.

(b) T. If γ arc-length parametrised. $T = \gamma'(t)$.

$K = \|\gamma''(t)\| = \|T'\| = 0$.

(c) F $\chi(\text{Torus}) = 0 \neq 76 - 92 + 100$

(d) F. From lectures K for cylinder $= 0$

(e) F $K = \lambda_1 \lambda_2 \neq 0$.

2(a) Define a path in S through s to be a smooth map $\gamma : (-\varepsilon, \varepsilon) \to S$ with $\gamma(-\varepsilon, \varepsilon) \subseteq S$ & $\gamma(0) = s$. Define

$$T_s S = \left\{ \gamma'(0) \,/\, \gamma \text{ is a smooth path in } S \text{ through } s \right\}$$

(b) Let $v \in \mathbb{R}^n$ and define $\gamma_v(t) = \psi(x + tv)$.

Then γ_v is a path in S through s as $\psi(U) \subseteq S$ & $\gamma_v(0) = \psi(x) = s$. So $\gamma_v'(0) \in T_s S$. But $\gamma_v'(0) = \psi'(x)(v)$ by the chain rule. So $\operatorname{im} \psi'(x) \subseteq T_s S$.

But $\psi'(x)$ is 1-1 as ψ is a parametrisation

Hence $\dim \operatorname{im} \Psi'(x) = n = \dim T_s S$. $\therefore$
$\operatorname{im} \Psi'(x) = T_s S$.

(c). We can define a parametrisation by
$\Psi(x,y) = (x, y, x^2+y^2)$. Then $\Psi'(x) =$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x & 2y \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{and has image}$$

$$T_{\Psi(x,y)}(S) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} \right\}.$$

3(a) Choose an arc-length parametrisation
$\tilde{\gamma} : (c,d) \to \mathbb{R}^3$ then

$$T = \tilde{\gamma}'$$

$$N = \frac{T'}{\|T'\|}$$

$$B = T \times N$$

(b) $\gamma(t) = (t, t, \frac{t^2}{2})$.

(i) $\gamma'(t) = (1, 1, t)$

$\|\gamma'(t)\| = \sqrt{2+t^2} \neq 1$ so not arc-length parametrised.

(ii)

$$T = (2+t^2)^{-\frac{1}{2}}(1,1,t)$$

$$T' = -\frac{1}{2}(2+t^2)(2t)(1,1,t) + (2+t^2)^{-\frac{1}{2}}(0,0,1)$$

$$= (2+t^2)^{3/2}\left[(-t)(1,1,t) + (2+t^2)(0,0,1)\right]$$

$$= (2+t^2)^{-3/2}((-t,-t,2))$$

$$N = \frac{T'}{\|T'\|} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2+t^2}}(-t,-t,2)$$

$$B = T \times N = \frac{1}{\sqrt{2}}\frac{1}{(2+t^2)}(2+t^2, -(2+t^2), 0)$$

$$= \frac{1}{\sqrt{2}}(1,-1,0).$$

(iii) $\dot{T} = \kappa N$, $\quad \dot{T} = \dfrac{T'}{\|\gamma'\|} = \dfrac{(-t,-t,2)}{(2+t^2)^2}$

$$= \frac{\sqrt{2}}{(2+t^2)^{3/2}} N$$

$$\dot{B} = 0 \quad \therefore \quad \tau = 0$$

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4 (a) $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\psi(x,y) = (x, y, x^2+y^2)$$

(b) $\frac{\partial \psi}{\partial x} = (1, 0, 2x)$ $\frac{\partial \psi}{\partial y} = (0, 1, 2y)$.

$$\frac{\partial \psi}{\partial x} \times \frac{\partial \psi}{\partial y} = \begin{bmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{bmatrix}$$

$$= (-2x, -2y, 1).$$

at (0,0,0) pointing up.

$$\therefore \quad n = \frac{(-2x, -2y, 1)}{\Delta^{\frac{1}{2}}} \qquad \underline{\Delta = 4x^2+4y^2+1}$$

(c) $$\frac{\partial n}{\partial x} = \frac{\partial}{\partial x}\left((4x^2+4y^2+1)^{-\frac{1}{2}} (-2x, -2y, 1)\right)$$

$$= \left(\left(-\tfrac{1}{2}\right) (\Delta)^{-3/2}\, 8x\, (-2x, -2y, 1) + \Delta^{-\frac{1}{2}}(-2,0,0)\right)$$

$$= \Delta^{-3/2}\left((-4x)(-2x, -2y, 1) + \Delta(-2, 0, 0)\right)$$

$$= \Delta^{-3/2}\left[(-8y^2-2), 8xy, -4x \right]$$

$$= \frac{2}{\Delta^{3/2}} \left[-(1+4y^2)\frac{\partial \psi}{\partial x} + 4xy \frac{\partial \psi}{\partial y} \right]$$

By symmetry get

$$\pi \doteq \frac{2}{\Delta^{3/2}} \begin{bmatrix} 1+4y^2 & -4xy \\ -4xy & 1+4y^2 \end{bmatrix}$$

$$\text{Mean curvature} = \frac{1}{2}\operatorname{tr}\pi = \frac{2+4x^2+4y^2}{(1+4x^2+4y)^{3/2}}$$

$$\text{Gaussian curvature} = \det\pi = \frac{4}{(1+4x^2+4y^2)^2}$$

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(5) (a) (i). Let $e^1, \ldots, e^n$ be ~~the~~ an orthonormal basis of $T_s S$. Then

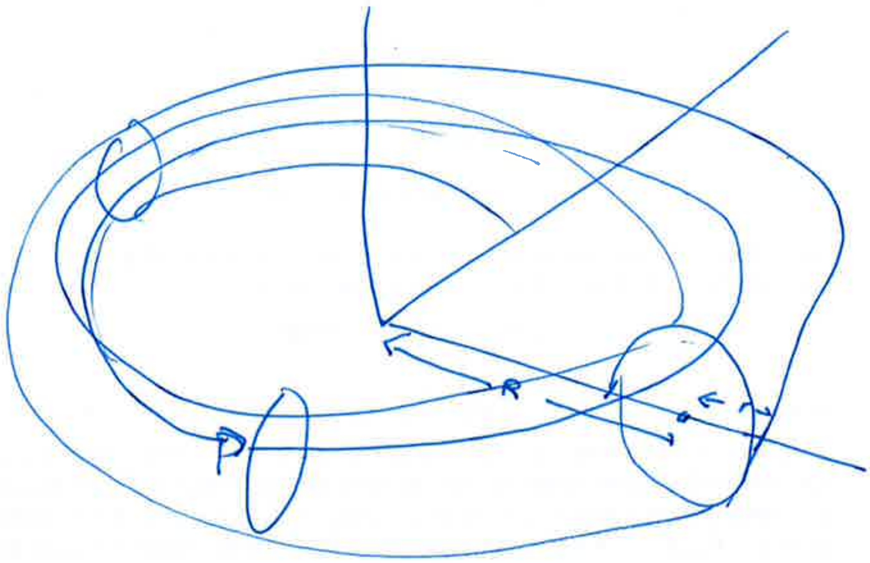
$$vd_S = \pm[e^1, \ldots, e^n]$$

where the sign is chosen so that vd_s is ~~a~~ positive.

(ii). We have

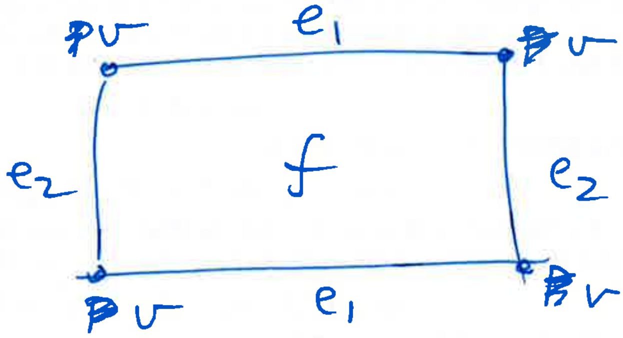
$$\int_{\psi(u)} f\, vd_S = \int_{u} f\circ\psi\ vd_S\left(\frac{\partial\psi}{\partial x^1}, \ldots, \frac{\partial\psi}{\partial x^n}\right) dx^1 \ldots dx^n.$$

(b). (i)



Torus.

Tessellate so that when cut open it looks like



$$\chi(T) = 1 - 2 + 1 = 0.$$

(ii). $$\frac{1}{2\pi} \int_T R \, v \, d_T = \chi(T).$$

$$= 0.$$