Geometry of Surfaces 2011

Mid-semester examination — Solutions

1.

(a) F. $f(x) = |x|$ is the standard example of a function which is continuous at 0 but not differentiable at 0.

(b) T. f is differentiable at a so $-f$ is differentiable at a so $g = (f+g) + (-f)$ is differentiable at a using standard results from lectures.

(c) T. Follows from the chain rule.

(d) T. If f is C^1 on $\mathbb{R}^3$ then it is differentiable on all of $\mathbb{R}^3$ from a result in lectures. Hence it is differentiable at $(1,2,3)$.

(e) F. The inverse $f^{-1}(x) = x^{1/3}$ is not infinitely differentiable. It is not differentiable once at 0.

[2 + 2 + 2 + 2 + 2 = 10]

2.

(a) Let U be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$. We say that f is differentiable at $a \in U$ if there is a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0.$$

(b) Let $h = (\alpha, \beta)$ then $f(a) = (1,1,1)$ and $f(a+h) = f(1+\alpha, 1+\beta) = (1+\alpha, 1+\alpha+\beta+\alpha\beta, 1+2\beta+\beta^2)$. Let $L(h) = (\alpha, \alpha+\beta, 2\beta)$ which is linear. Then

$$\|f(a+h) - f(a) - L(h)\| = \|(0, \alpha\beta, \beta^2)\| \leq \sqrt{\alpha^2+\beta^2+\beta^4} \leq \sqrt{(\alpha^2+\beta^2)^2} \leq \|h\|^2.$$

This

$$0 \leq \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} \leq \|h\| \to 0$$

as $h \to 0$. Hence by the Squeeze Lemma

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0.$$

and f is differentiable at $(1,1)$. We have $f'(1,1)(h) = (\alpha, \alpha+\beta, 2\beta)$.

[4 + 6 = 10]

3.

(a) A subset $S \subseteq \mathbb{R}^N$ is called a *submanifold* of dimension n if for all $s \in S$ there exists a U open in $\mathbb{R}^N$, containing s, and a smooth map $\phi\colon U \to \mathbb{R}^n$ such that $\phi(U)$ is open, $\phi\colon U \to \phi(U)$ is a diffeomorphism and

$$S \cap U = \{x \in U \mid (\phi^{n+1}(x), \ldots, \phi^N(x)) = 0\}.$$

(b) Consider $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$. Then $S^2 = F^{-1}(0)$ where $F(x,y,z) = x^2+y^2+z^2-1$. The function $F\colon \mathbb{R}^3 \to \mathbb{R}$ is a polynomial so smooth, and is a defining equation for S^2 because $F'(x,y,z) = (2x, 2y, 2z) \neq 0$ for every $(x,y,z) \in S^2$ and hence, from results in lectures, S^2 is a submanifold of dimension two and thus a surface.

(c) Define $f\colon S^2 \to \mathbb{R}$ by $f(x,y,z) = z$. This is the restriction of $\tilde{f}\colon \mathbb{R}^3 \to \mathbb{R}$ $\tilde{f}(x,y,z) = z$ which is clearly smooth so that f is smooth.

[3 + 4 + 3 = 10]