School of Mathematical Sciences
PURE MTH 3022
Geometry of Surfaces III, Semester 2, 2011

Summary of the course. Wednesday November 2nd.

Contents

1. Review

We have the usual notation $\mathbb{R}$ for the real numbers, $\mathbb{R}^n$ for n-tuples of real numbers and $\mathbb{N} = \{1, 2, 3, \dots\}$ for the natural numbers. Recall that if $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^n)$ are in $\mathbb{R}^n$ then $\|x\| = \sqrt{\sum_{i=1}^n (x^i)^2}$ and $\langle x, y\rangle = \sum_{i=1}^n x^i y^i$. These satisfy

$$\langle x, y\rangle \leq \|x\|\|y\| \qquad \textbf{Cauchy's inequality},$$
$$\|x + y\| \leq \|x\| + \|y\| \qquad \textbf{Triangle inequality}$$

and

$$\max\{|x^1|, \dots, |x^n|\} \leq \|x\| \leq \sqrt{n}\max\{|x^1|, \dots, |x^n|\}.$$

Note 1.1. Those who have done Real Analysis or Topology and Analysis will recall that $\mathbb{R}^n$ with the *Euclidean metric* $d(x,y) = \|x-y\|$ is a metric space. We don't need the metric notion for this course but we will use many of the other notions of metric spaces such as open balls, open sets, sequences, limits and continuous functions but usually only for the metric space $\mathbb{R}^n$. We will only need sequences briefly to prove the Contraction Mapping Theorem.

The open ball around $x \in \mathbb{R}^n$ of radius $\epsilon > 0$ is $B(x,\epsilon) = \{y \in \mathbb{R}^n \mid \|x-y\| < \epsilon\}$ and a subset $U \subseteq \mathbb{R}^n$ is called *open* if for every $x \in U$ there is some $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$.

A *sequence* in $\mathbb{R}^n$ is a function $\mathbb{N} = \{1,2,\dots\} \to \mathbb{R}^n$ usually denoted by its set of values $x_1, x_2, \dots$ or $(x_n)_{n=0}^{\infty}$ or often just (x_n). A sequence (x_n) has *limit* $x \in \mathbb{R}^n$ if for all $\epsilon > 0$ there is an N such that for all $n \geq N$ we have $\|x_n - x\| < \epsilon$. In such a case we also say that x_n converges to x and write $\lim_{n\to\infty} x_n = x$ or just $x_n \to x$.

[2011: End of Lecture 1]

Lemma 1.1. *If (x_n) has a limit it is unique.*

Lemma 1.2. *A sequence (x_n) has* limit *$x \in \mathbb{R}^n$ if and only if $\lim_{n\to\infty} \|x_n - x\| = 0$.*

Lemma 1.3 (Squeeze Lemma)**.** *Let (x_n), (y_n) and (z_n) be sequences of real numbers with $x_n \leq y_n \leq z_n$ for all n. If $x_n \to x$ and $z_n \to x$ then $y_n \to x$.*

Proposition 1.4 (Properties of limits of sequences)**.**

(1) If $x_m = (x_m^1, \dots, x_m^n) \in \mathbb{R}^n$ then $\lim_{m\to\infty} x_m = x = (x^1, \dots, x^n)$ if and only if $\lim_{m\to\infty} x_m^i = x^i$ for all $i = 1, \dots, n$.
(2) If $x_n \to x$ and $y_n \to y$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha x_n + \beta y_n \to \alpha x + \beta y$.

Lemma 1.5. *If $x_n \to x$ then $\lim_{m,n\to\infty} \|x_n - x_m\| \to 0$.*

A sequence with $\lim_{m,n\to\infty} \|x_n - x_m\| \to 0$ is called *Cauchy*.

Theorem 1.6. *Every Cauchy sequence in $\mathbb{R}^n$ converges.*

[2011: End of Lecture 2]

Definition 1.7. Let $a \in U \subseteq \mathbb{R}^n$, U open and $f\colon U - \{a\} \to \mathbb{R}^m$. We say that f has limit L at a if $\forall \epsilon > 0\ \exists \delta > 0$ such that if $\|x - a\| < \delta$ and $x \in U$ then $\|f(x) - L\| < \epsilon$.

If f has a limit L at a it is unique and we write $\lim_{x\to a} f(x) = L$.

Proposition 1.8. *A function $f\colon U - \{a\} \to \mathbb{R}^m$ has limit L at a if and only if for all sequences $(x_n) \subseteq U - \{a\}$ with $x_n \to a$ we have $f(x_n) \to L$.*

Lemma 1.9 (Squeeze Lemma)**.** *Let $f, g, h\colon U - \{a\} \to \mathbb{R}$ be functions with $f(x) \leq g(x) \leq h(x)$ for all $x \in U - \{a\}$. If $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = L$.*

Proposition 1.10 (Properties of limits)**.**

(1) Let $f\colon U - \{a\} \to \mathbb{R}^m$ and let $f(x) = (f^1(x), \dots, f^m(x)$ where $f^i\colon U - \{a\} \to \mathbb{R}$ for each $i = 1, \dots m$. Then $\lim_{x\to a} f(x) = f(a)$ if and only if for every $i = 1, \dots m$ we have $\lim_{x\to a} f^i(x) = f^i(a)$.
(2) Let $f, g\colon U - \{a\} \to \mathbb{R}^m$ with $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = J$. If $\alpha, \beta \in \mathbb{R}$ then $\lim_{x\to a} \alpha f(x) + \beta g(x) = \alpha L + \beta J$.

Definition 1.11. Let U be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$. We say that f is *continuous* at $a \in U$ if $\lim_{x\to a} f(x) = f(a)$ and we say that f is *continuous* on U if f is continuous at every $a \in U$.

Proposition 1.12. *A function f is continuous at a if and only if $\lim_{x\to a} \|f(x) - f(a)\| = 0$.*

Proposition 1.13. *A function $f\colon U \to \mathbb{R}^m$ is continuous at a if and only if for every sequence with $x_n \to a$ we have $f(x_n) \to f(a)$.*

[2011: End of Lecture 3]

Proposition 1.14 (Properties of continuous functions)**.**

(1) If U is open in $\mathbb{R}^n$ and $f, g\colon U \to \mathbb{R}^m$ are continuous and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g$ is continuous.
(2) If U is open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$ then $f = (f^1, \dots, f^m)$ is continuous if and only if each $f^i\colon U \to \mathbb{R}$ is continuous for every $i = 1, \dots, m$.
(3) If $f\colon U \to \mathbb{R}^m$ and $g\colon V \to \mathbb{R}^k$ and U is open in $\mathbb{R}^n$ and V is open in $\mathbb{R}^m$ and $f(U) \subseteq V$ then f and g continuous implies that $g \circ f$ is continuous.

Let $X \subseteq \mathbb{R}^n$. Recall that $f\colon X \to X$ is called a *contraction* if there exists $0 \le K < 1$ such that for all $x, y \in X$ we have $\|f(x) - f(y)\| \le K\|x - y\|$.

Proposition 1.15 (Contraction mapping theorem)**.** *If $f\colon \overline{B}(0, r) \to \overline{B}(0, r)$ is a contraction then there is a unique $x \in \overline{B}(0, R)$ such that $f(x) = x$.*

Note 1.2. Those of you who have done Real Analysis will know that the Contraction Mapping Theorem is usually proved for a contraction on a complete metric space. That more general result reduces to this case as $\overline{B}(0, r)$ is a closed subset of the complete metric space $\mathbb{R}^n$ and hence complete.

2. Differentiation in $\mathbb{R}^n$.

Definition 2.1. Let U be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$. We say that f is differentiable at $a \in U$ if there is a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\|f(a + h) - f(a) - L(h)\|}{\|h\|} = 0.$$

Lemma 2.2. *If f is differentiable at a then the L in the definition is unique.*

If f is differentiable at a we denote the linear map L by $f'(a)$. If f is differentiable at every $a \in U$ we say that f is differentiable on U.

[2011: End of Lecture 4]

Proposition 2.3. *Let U be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$. Define $f^i\colon U \to \mathbb{R}$ for $i = 1, \dots, m$ by $f(x) = (f^1(x), \dots, f^m(x)$ for all $x \in U$. Then f is differentiable at $a \in U$ if and only if each of the f^i is differentiable at a and $f'(a) = (f^{1'}(a), \dots, f^{m'}(a))$.*

Lemma 2.4. *The function f is differentiable at a if and only if there exists a linear function L, an $\epsilon > 0$ and a function $R\colon B(0, \epsilon) \to \mathbb{R}^m$ such that $f(a + h) = f(a) + L(h) + R(h)$ and $\lim_{h \to 0} \|R(h)\|/\|h\| = 0$.*

Proposition 2.5. *If $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and $v \in \mathbb{R}^m$ then $f(x) = L(x) + v$ is differentiable on all of $\mathbb{R}^n$ and $f'(x) = L$.*

Proposition 2.6. *If $f\colon U \to \mathbb{R}^m$ is differentiable at a then f is continuous at a.*

Lemma 2.7. *If f is differentiable at a then $\forall \epsilon > 0$ $\exists \delta > 0$ such that if $\|h\| < \delta$ then $\|f(a + h) - f(h)\| \le (\|f'(a)\| + \epsilon)\|h\|$.*

Proposition 2.8. *If $f, g\colon U \to \mathbb{R}^m$ are differentiable at $a \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g$ is differentiable at a and $(\alpha f + \beta g)'(a) = \alpha f'(a) + \beta g'(a)$.*

[2011: End of Lecture 5]

Proposition 2.9. *If $f, g\colon U \to \mathbb{R}$ are differentiable at $a \in \mathbb{R}^n$ then fg is differentiable at a and $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$.*

Proposition 2.10 (Chain Rule)**.** *Let $f\colon U \to \mathbb{R}^m$ and $g\colon V \to \mathbb{R}^k$ where U is open in $\mathbb{R}^n$ and V is open in $\mathbb{R}^m$ with $f(U) \subseteq V$. Then if f is differentiable at a and g is differentiable at $f(a)$ then $g \circ f$ is differentiable at a and $(g \circ f)'(a) = g'(f(a)) \circ f'(a)$.*

[2011: End of Lecture 6]

Proposition 2.11. *If $U \subseteq \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}^m$ is differentiable at a and $v \in \mathbb{R}^n$ then*

$$f'(a)(v) = \frac{d}{dt} f(a+tv)\Big|_{t=0}.$$

Corollary 2.12. *Let f be as above and write $f = (f^1, \dots, f^m)$. Let e^i be the vector with a 1 in the ith place and zeros elsewhere. Then*

$$f'(a)(e^i) = \left(\frac{\partial f^1}{dx^i}(a), \dots, \frac{\partial f^m}{\partial x^i}(a)\right).$$

so that the linear map $f'(a)\colon \mathbb{R}^n \to \mathbb{R}^m$ is the Jacobian matrix

$$J(f)(a) = \frac{\partial f^i}{\partial x^j}(a).$$

2.1. **Functions of class C^k.**

Definition 2.13. Let $f\colon U \to \mathbb{R}$ for U open in $\mathbb{R}^n$. We say that f is (of class) C^k if all partial derivatives of f exist and are continuous on U up to and including order k. We write C^0 for continuous functions and C^∞ or *smooth* for functions which are in C^k for every k. The set of all C^k functions on U is denoted by $C^k(U)$ or $C^k(U, \mathbb{R})$.

Let $f\colon U \to \mathbb{R}^m$ for U open in $\mathbb{R}^n$ and let $f = (f^1, \dots, f^m)$. We say that f is C^k if each $f^i\colon U \to \mathbb{R}$ is C^k for $i = 1, \dots, m$. Again we write $C^k(U, \mathbb{R}^m)$ for the set of all such f.

Lemma 2.14. *Let $a < b < c$ and assume that $f\colon (a,c) \to \mathbb{R}$ is continuous and $f'\colon (a,b) \cup (b,c) \to \mathbb{R}$ is continuous and*

$$\alpha = \lim_{t \to b^-} f'(x) = \lim_{t \to b^+} f'(x)$$

then f is C^1 on (a,c) and $f'(b) = \alpha$.

[2011: End of Lecture 7]

Proposition 2.15. *If $f \in C^1(U)$ then f is differentiable at a for all $a \in U$.*

Proposition 2.16. *If $f \in C^2(U)$ then*

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$

on U. Similarly if f is C^k for $k \geq 2$ then all partial derivatives up to order including k are independent of order.

Proposition 2.17. *$C^k(U, \mathbb{R}^m)$ is a vector space.*

Proposition 2.18 (Chain Rule)**.** *Let $f\colon U \to \mathbb{R}^m$ and $g\colon V \to \mathbb{R}^k$ where U is open in $\mathbb{R}^n$ and V is open in $\mathbb{R}^m$ with $f(U) \subseteq V$. Assume that f and g are C^1 then*

$$\frac{\partial (g \circ f)^j}{\partial x^i}(a) = \sum_{l=1}^{m} \frac{\partial g^j}{\partial x^l}(f(a)) \frac{\partial f^l}{\partial x^i}(a).$$

for every $i = 1, \dots, n$ and $j = 1, \dots, k$.

[2011: End of Lecture 8]

2.2. **Mean Value Theorem.**

For any x_0 and x_1 in $\mathbb{R}^n$ we define $[x_0, x_1]$ to be the line segment joining x_0 to x_1 that is $[x_0, x_1] = \{(1-t)x_0 + tx_1 \mid t \in [0,1]\}$.

Proposition 2.19 (Mean Value Theorem)**.** *If U is open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$ is differentiable and $[x_0, x_1] \subseteq U$ and $u \in \mathbb{R}^m$ then $\exists \xi_u \in [x_0, x_1]$ such that*

$$\langle f(x_1), u \rangle = \langle f(x_0), u \rangle + \langle f'(\xi_u)(x_1 - x_0), u \rangle.$$

Corollary 2.20. *If U is open in $\mathbb{R}^n$ and $h\colon U \to \mathbb{R}^m$ is differentiable and $\|h'(\xi)\| \leq \epsilon \ \forall \xi \in [x_0, x_1]$ then $\|h(x_0) - h(x_1)\| < \epsilon \|x_0 - x_1\|$.*

2.3. **Inverse Function Theorem.**

Theorem 2.21 (Inverse Function Theorem)**.** *Let U be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^n$ be C^k for $k \geq 1$. Assume that $f'(a)$ is invertible for some $a \in U$. Then there is an open set $V \subseteq U$ with $a \in V$ such that:*
(1) $f(V)$ is open,
(2) $f\colon V \to f(V)$ is invertible,
(3) f^{-1} is C^k, and
(4) $(f^{-1})'(f(a)) = [f'(a)]^{-1}$.

[2011: End of Lecture 9]

[2011: End of Lecture 10]

Corollary 2.22 (Open mapping theorem)**.** *Let U be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^n$ be such that $f'(x)$ is invertible for all $x \in U$. Then $f(U)$ is open in $\mathbb{R}^n$.*

Definition 2.23. If U and V are open in $\mathbb{R}^n$ and $f\colon U \to V$ is C^k with a C^k inverse then f is called a C^k *diffeomorphism*.

For the implicit function theorem we need the following notation. If $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ then we denote by (x, y) the obvious element $\mathbb{R}^n \times \mathbb{R}^m$.

Theorem 2.24 (Implicit Function Theorem)**.** *Let U be open in $\mathbb{R}^{n+m}$, $(x_0, y_0) \in U$ and $F\colon U \to \mathbb{R}^m$ be C^k. If $F(x_0, y_0) = 0$ and*

$$\frac{\partial F}{\partial y}(x_0, y_0) = \begin{pmatrix} \frac{\partial F^1}{\partial y^1}(x_0, y_0) & \dots & \frac{\partial F^1}{\partial y^m}(x_0, y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial y^1}(x_0, y_0) & \dots & \frac{\partial F^m}{\partial y^m}(x_0, y_0) \end{pmatrix}$$

is non-singular then there exists an open set $\tilde{V} \subseteq \mathbb{R}^{n+m}$ containing (x_0, y_0) and a C^k function $f\colon V \to \mathbb{R}^m$, where $V = \{x \in \mathbb{R}^n \mid (x, 0) \in \tilde{V}\}$ such that

$$\tilde{V} \cap \{(x, y) \mid F(x, y) = 0\} = \{(x, f(x)) \mid x \in V\}.$$

3. Submanifolds

Definition 3.1. A subset $S \subseteq \mathbb{R}^N$ is called a *submanifold* of dimension n if for all $s \in S$ there exists a U open in $\mathbb{R}^N$, containing s, and a smooth map $\phi\colon U \to \mathbb{R}^N$ such that $\phi(U)$ is open, $\phi\colon U \to \phi(U)$ is a diffeomorphism and

$$S \cap U = \{x \in U \mid (\phi^{n+1}(x), \dots, \phi^N(x)) = 0\}.$$

[2011: End of Lecture 11]

Theorem 3.2. *Let $S \subseteq \mathbb{R}^N$ then the following are equivalent.*
(1) S is a submanifold of dimension n;
(2) for every $s \in S$ there is an open set $U \subseteq \mathbb{R}^N$ containing s and a smooth function $F\colon U \to \mathbb{R}^{N-n}$ such that $F'(x)$ is onto for all $x \in S \cap U$ and $S \cap U = \{x \in U \mid F(x) = 0\}$;
(3) for all $s \in S$ there exists V open in $\mathbb{R}^N$ such that $S \cap V$ is the graph of a smooth function of n of the N variables;
(4) for all $s \in S$ there is a V open in $\mathbb{R}^N$ containing s and U open in $\mathbb{R}^n$ and an $f\colon U \to V$ which is one to one with $f'(x)$ one to one for all $x \in U$ and such that $S \cap V = f(U)$.

[2011: End of Lecture 12]

Corollary 3.3. *If U is open in $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^{N-n}$ is smooth with $F'(x)$ onto for all $x \in S = F^{-1}\{0\}$ then S is a submanifold of dimension n.*

Definition 3.4. Let $S \subseteq \mathbb{R}^N$ be an n dimensional submanifold. If U is an open set in $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^{N-n}$ is smooth with $F'(s)$ onto for all $s \in S \cap U$ and $S \cap U = \{s \in U \mid F(s) = 0\}$ then F is called a *local defining equation* for S.

Definition 3.5. Let $S \subseteq \mathbb{R}^N$ be an n dimensional submanifold. If U is an open subset of $\mathbb{R}^n$ and V an open subset of $\mathbb{R}^N$ and $\psi\colon U \to V$ is smooth and one to one with $\psi'(x)$ one to one for all $x \in U$ and $\psi(U) = S \cap V$ then ψ is called a *local parametrisation* of S.

3.1. **Tangent space to a submanifold.**

Definition 3.6. Let s be a point in a submanifold $S \subseteq \mathbb{R}^N$. Let $\epsilon > 0$. Then a smooth map $\gamma\colon (-\epsilon, \epsilon) \to S \subseteq \mathbb{R}^N$ with $\gamma(0) = s$ is a called a *smooth path in S through s*.

[2011: End of Lecture 13]

Definition 3.7. Define T_sS to be the union of all the vectors $\gamma'(0)$ for γ a smooth path in S through s. Call it the *tangent space to S at s*.

Proposition 3.8. *T_sS is an n-dimensional subspace of $\mathbb{R}^N$.*

Proposition 3.9. *If F is a local defining equation for S defined on an open set containing s then $T_sS = \ker F'(s)$. If ψ is a local parametrisation for S with $\psi(x) = s$ then $T_sS = im\, \psi'(x)$. Moreover $\frac{\partial \psi}{\partial x^1}(x), \dots, \frac{\partial \psi}{\partial x^n}(x)$ are a basis for T_sS.*

[2011: End of Lecture 14]

3.2. **Smooth functions on submanifolds.** Let $S \subseteq \mathbb{R}^N$ be a submanifold and let $f\colon S \to \mathbb{R}$ be a function.

Definition 3.10. We say that f is a smooth function if for all $s \in S$ there is an open set $U \subseteq \mathbb{R}^N$ with $s \in U$ and a smooth function $\tilde{f}\colon U \to \mathbb{R}$ such that $f_{|S\cap U} = \tilde{f}_{S\cap U}$.

Proposition 3.11. *Let $S \subseteq \mathbb{R}^N$ be a submanifold and $f\colon S \to \mathbb{R}$ a function.*
(1) If $\psi\colon U \to S$ is a parametrisation and f is smooth then $f \circ \psi\colon U \to \mathbb{R}$ is smooth.
(2) If for every $s \in S$ there is a parametrisation $\psi\colon U \to S$ with $s \in \psi(U)$ such that $f \circ \psi\colon U \to \mathbb{R}$ is smooth then f is smooth.

[2011: End of Lecture 15]

Proposition 3.12. *Let $S \subseteq \mathbb{R}^N$ be a submanifold and $\psi\colon U \to S$ and $\chi\colon V \to S$ be parametrisations with $\psi(U) = \chi(V)$ then $\psi^{-1} \circ \chi\colon V \to U$ is a diffeomorphism.*

[2011: End of Lecture 16]

Let S be a submanifold and $f\colon S \to \mathbb{R}$ be a smooth function.

Proposition 3.13. *Let $U \subseteq \mathbb{R}^N$ be open and $g\colon U \to \mathbb{R}^N$ be a smooth function with image $g(U)$ inside a smooth submanifold $S \subseteq \mathbb{R}^N$. If $f\colon S \to \mathbb{R}^m$ is smooth then $f \circ g\colon U \to \mathbb{R}^m$ is smooth.*

Proposition 3.14. *Let $S \subseteq \mathbb{R}^N$ be a smooth submanifold and $f\colon S \to \mathbb{R}^m$ be a smooth function. If $\gamma_1, \gamma_2\colon (-\epsilon, \epsilon) \to S$ be two smooth paths through $s \in S$ with $\gamma_1'(0) = \gamma_2'(0)$ then*

$$(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0).$$

If $f\colon S \to \mathbb{R}^m$ is smooth and $v \in T_sS$ we define $f'(s)(v) \in \mathbb{R}^m$ by choosing a smooth path γ through s with $\gamma'(0) = v$ and letting $f'(s)(v) = (f \circ \gamma)'(0)$.

Proposition 3.15. *The function $f'(s)\colon T_sS \to \mathbb{R}^m$ is well-defined and linear. Moreover if $\tilde{f}$ is an extension of f then $f'(s) = \tilde{f}'(s)_{|T_sS}$.*

Proposition 3.16. *Let $S \subseteq \mathbb{R}^N$ be a submanifold and $f\colon S \to \mathbb{R}^m$ be smooth. If $g\colon U \to \mathbb{R}^N$ is smooth with $g(U) \subseteq S$ then*

$$(f \circ g)'(x) = f'(g(x)) \circ g'(x).$$

4. Geometry of curves

Definition 4.1. A curve is a one-dimensional submanifold.

If c is a point in a curve C then T_cC is one-dimensional so that $T_cC - \{0\}$ has two connected components. A continuous choice of one of these two components at each point of C is called an *orientation* and a curve with an orientation is called an *oriented curve*.

Definition 4.2. A parametrised curve C is a curve for which there is a parametrisation $\gamma\colon (a,b) \to C$ with $\gamma(a,b) = C$.

For a parametrisation $\gamma'(t) \neq 0$. If $\gamma'(t)$ is in the chosen half of $T_{\gamma(t)}C$ for an oriented curve C then we say the parametrisation is oriented.

Definition 4.3. We say a parametrised curve is parametrised by arc length if $\|\gamma'(t)\| = 1$ for all t.

[2011: End of Lecture 17]

Proposition 4.4. *If $\gamma\colon (a,b) \to C$ is a parametrised curve then it has a parametrisation by arc-length.*

Lemma 4.5. *If $\gamma(t)$ and $\tilde{\gamma}(t)$ are two arc length parametrisations of a curve C then there is a $t_0 \in \mathbb{R}$ such that $\gamma(t) = \tilde{\gamma}(t + t_0)$ for all t.*

Lemma 4.6. *If $\gamma(t)$ is parametrised by arc length then $\langle \gamma''(t), \gamma'(t)\rangle = 0$.*

Definition 4.7. If C is a curve with an arc length parametrisation $\gamma(t)$ then the *curvature of C* at $c = \gamma(t)$ is $\kappa(c) = \|\gamma''(t)\|$.

[2011: End of Lecture 18]

Proposition 4.8. *If C is a curve and $\gamma(t)$ is a (not necessarily arc-length) parametrisation then*

$$\begin{aligned}\kappa &= \frac{1}{\|\gamma'\|^2}\left\|\gamma'' - \gamma'\frac{\langle\gamma',\gamma''\rangle}{\|\gamma'\|^2}\right\| \\ &= \frac{1}{\|\gamma'\|^2}\left(\|\gamma''\|^2 - \frac{\langle\gamma',\gamma''\rangle^2}{\|\gamma'\|^2}\right)^{1/2}\end{aligned}$$

4.1. Curves in $\mathbb{R}^3$.

Definition 4.9. Let $\gamma\colon (a,b) \to C$ be a curve in $\mathbb{R}^3$ parametrised by arc-length. Let $c = \gamma(t)$. We define

(1) $T(c) = \gamma'(t)$ the *unit tangent vector* at c;
(2) $N(c) = T'(c)/\|T'(c)\|$ the *principal unit normal* at c; and
(3) $B(c) = T(c) \times N(c)$ the unit binormal at c.

$T(c), N(c)$ and $B(c)$ define an orthonormal basis for $\mathbb{R}^3$ for each $c \in C$.

Proposition 4.10. *Let $\gamma\colon (a,b) \to C$ be a parametrised curve and let $c = \gamma(t)$.*

(1) $T(c) = \gamma'(t)/\|\gamma'(t)\|$ and
(2) $N(c) = T'(c)/\|T'(c)\|$.

Proposition 4.11 (Frenet formula). *Let $\dot{T}$, $\dot{N}$ and $\dot{B}$ denote differentiation with respect to arc-length. Then we have*

$$\dot{T} = \kappa N \qquad \dot{N} = -\kappa T + \tau B \qquad \dot{B} = -\tau N$$

for a function τ on the curve called the torsion *of the curve.*

[2011: End of Lecture 19]

5. Geometry of surfaces

Let $\Sigma \subseteq \mathbb{R}^3$ be a surface. Then $T_s\Sigma^\perp$, the orthogonal space to the tangent space $T_s\Sigma$, is one-dimensional so if zero is removed there are two connected halves. In other words there are two possible unit normals. An orientation for Σ is a choice of unit normal $n(s) \in T_s\Sigma^\perp$ continuously across the surface. An oriented surface is a surface with an orientation. If $\psi\colon U \to \Sigma$ is a parametrisation we say it is oriented if

$$n = \frac{\frac{\partial\psi}{\partial x^1} \times \frac{\partial\psi}{\partial x^2}}{\left\|\frac{\partial\psi}{\partial x^1} \times \frac{\partial\psi}{\partial x^2}\right\|}.$$

The unit normal defines a map $n\colon \Sigma \to S^2$ called the *Gauss map*.

If $\psi\colon U \to \Sigma$ are local parameters for a surface Σ with $\psi(x) = s$ and v and w are in $T_s\Sigma$ define the *second fundamental form* α by

$$\alpha(s)(v,w) = \sum_{i,j=1}^{2} v_i w_j \left\langle \frac{\partial^2\psi}{\partial x^i \partial x^j}(x), n \right\rangle$$

where $v = \sum_{i=1}^2 v_i \frac{\partial\psi}{\partial x^i}$ and $w = \sum_{i=1}^2 w_i \frac{\partial\psi}{\partial x^i}$.

[2011: End of Lecture 20]

Proposition 5.1. *Let α be the second fundamental form at a point s of a surface Σ then:*
(1) $\alpha(v,w) = -\langle n'(v), w)\rangle = -\langle n'(w), v\rangle$,
(2) α is independent of the parametrisation.

Note that here $n'(v) = n'(s)(v)$.

Lemma 5.2. *Let $s \in \Sigma$ and let $v \in T_s\Sigma$. Let the unit normal at s be n. Then $\exists \epsilon > 0$ and a map $a\colon (-\epsilon,\epsilon) \to \Sigma$ with $a(0) = 0$ and $a'(0) = 0$ and $\gamma(t) = s + tv + a(t)n$ a curve in S.*

Proposition 5.3. *If Σ is a surface, $s \in \Sigma$, n the unit normal at s and $v \in T_s\Sigma$ then the intersection of $s + T_s\Sigma$ and Σ near s is a curve whose curvature is $\alpha(v,v)/\|v\|^2$.*

The first fundamental form at $s \in \Sigma$ is the inner product $g(v,w) = \langle v,w\rangle$.

[2011: End of Lecture 21]

Definition 5.4. Define a function $\Pi(s)\colon T_s\Sigma \to T_s\Sigma$ by $\Pi = -n'$.

Clearly $\alpha(v,w) = \langle \Pi(v), w\rangle$. Π is symmetric so it has orthogonal eigenvectors v_1 and v_2 with eigenvalues λ_1 and λ_2.

Definition 5.5. Let Π be as above.

(1) The eigenvalues λ_1, λ_2 of Π are called the *principal curvatures*.
(2) Their average $(1/2)\,\mathrm{tr}(\Pi)$ is called the *mean curvature*
(3) Their product $\det(\Pi)$ is called the *Gaussian curvature*.

Proposition 5.6. *If v_1 and v_2 are a basis for $T_s\Sigma$ and $\alpha_{ij} = \alpha(v_i,v_j)$ and $g_{ij} = g(v_i,v_j) = \langle v_i,v_j\rangle$ then $\Pi = \alpha g^{-1}$ so that $\det(\Pi) = \det(\alpha_{ij})/\det(g_{ij})$.*

[2011: End of Lecture 22]

6. Integration

6.1. **Integration in $\mathbb{R}^n$.** Let R be a closed bounded subset of $\mathbb{R}^n$ and $f\colon R \to \mathbb{R}$ a continuous function. Recall that we can defined an integral $\int_R f dx^1 \dots dx^n$.

Proposition 6.1. *The integral satisfies:*
(i) $\int_R f dx^1 \dots dx^n$ *is linear in* f,
(ii) if $f(x) \geq 0$ *for all* $x \in R$ *then* $\int_R f dx^1 \dots dx^n \geq 0$
(iii) if $f(x) \geq 0$ *then* $\int_R f dx^1 \dots dx^n$ *is the volume in* $\mathbb{R}^{n+1}$ *of the region consisting of all* $(x^1, \dots, x^{n+1})$ *such that* $(x^1, \dots, x^n) \in R$ *and* $0 \leq x^{n+1} \leq f(x, y)$ *and*
(iv) if R_1 *and* R_2 *are two regions with* $R_1 \cap R_2 = \emptyset$ *and* $R = R_1 \cup R_2$ *then* $\int_{R_1} f dx^1 \dots dx^n + \int_{R_2} f dx^1 \dots dx^n = \int_R f dx^1 \dots dx^n$.

We calculate using Fubini's theorem which we start in the case of $\mathbb{R}^2$.

Theorem 6.2 (Fubini)**.** *Let* R *be a closed bounded region in* $\mathbb{R}^2$ *and* $f: R \to \mathbb{R}$ *a continuous function. Then*

$$\int \left(\int f(x, y) dx \right) dy = \int_R f dx dy = \int \left(\int f(x, y) dy \right) dx.$$

Definition 6.3. Let U be an open subset of $\mathbb{R}^n$ and $f: U \to \mathbb{R}$ be continuous and define the *support* of f ($\operatorname{supp}(f)$) to be the closure in U of $\{x \in U \mid f(x) \neq 0\}$.

Theorem 6.4 (Change of variable formula)**.** *Let* U *and* V *be open subsets of* $\mathbb{R}^n$ *and let* $\rho: U \to V$ *be a diffeomorphism. Let* $f: V \to \mathbb{R}$ *have support which is closed and bounded in* $\mathbb{R}^n$ *then*

$$\int_U f dx^1 \dots dx^n = \int_V f \circ \rho |\det(J(\rho))| dx^1 \dots dx^n$$

where

$$J(\rho) = \begin{pmatrix} \frac{\partial \rho^1}{\partial x^1} & \cdots & \frac{\partial \rho^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \rho^n}{\partial x^1} & \cdots & \frac{\partial \rho^n}{\partial x^n} \end{pmatrix}$$

is the Jacobian matrix of ψ.

6.2. **Volume forms and integration.** Let V be a vector space of dimension n. An n-form is a multilinear and totally antisymmetric map

$$\omega: \underbrace{V \times \cdots \times V}_{\text{n times}} \to \mathbb{R}.$$

Multilinear means linear in each of the n factors separately, that is for any $w_i, v_1, \dots, v_n \in V$ and $a, b \in \mathbb{R}$ we have

$$\omega(v_1, \dots, av_i + bw_i, \dots, v_n) = a\omega(v_1, \dots, v_i, \dots, v_n) + b\omega(v_1, \dots, w_i, \dots, v_n).$$

Totally antisymmetric means that if π is a permutation of the numbers $1, \dots, n$ with sign[1] denoted by $\operatorname{sign}(\pi)$ then

$$\omega(v_{\pi(1)}, \dots, v_{\pi(n)}) = \operatorname{sign}(\pi)\omega(v_1, \dots, v_n).$$

Notice that if ω is an n-form then

$$\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

and hence

$$\omega(v_1, \dots, v, \dots, v, \dots, v_n) = 0.$$

If $v_1, \dots, v_n$ is a basis of V we define an n-form $[v_1, \dots, v_n]$ by

$$[v_1, \dots, v_n](w_1, \dots, w_n) = \det(X_{ij})$$

where $w_i = \sum_j^n X_{ij} v_j$ for each $i = 1, \dots, n$.

We denote the set of all n-forms by $\det(V^*)$. It is a vector space and we have

Proposition 6.5. *The space of all* n*-forms,* $\det(V^*)$*, is one dimensional. If* ω *is an* n*-form and* $v_1, \dots, v_n$ *is a basis then* $\omega = \omega(v_1, \dots, v_n)[v_1, \dots, v_n]$.

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[1] If $\hat{\pi}$ is the matrix whose (i, j)th entry is 1 if $\pi(i) = j$ and zero otherwise then $\operatorname{sign}(\pi) = \det(\hat{\pi})$.

Corollary 6.6. *Let $w_1, \dots, w_n$ and $v_1, \dots, v_n$ be bases of V. Let $w_i = \sum_{j=1}^n X_{ij} v_j$, then*

$$[v_1, \dots, v_n] = \det(X_{ij})[w_1, \dots, w_n].$$

If $\Sigma \subseteq \mathbb{R}^N$ is an n dimensional submanifold then each tangent space $T_s\Sigma$ is n dimensional and we can form $\det(T_s\Sigma^*)$ the space of all n-forms on $T_s\Sigma$. As $\det(T_s\Sigma^*)$ is one dimensional it follows that $\det(T_s\Sigma^*) - \{0\}$ has two connected components. We call Σ oriented if we have picked one of these two components at each s on Σ in a continuous manner. If ω is in the chosen half of $\det(T_s\Sigma^*) - \{0\}$ then we call it positive. If $\psi\colon U \to \Sigma$ is a parametrisation then we say it is oriented if the n form $[\frac{\partial\psi}{\partial x^1}, \dots, \frac{\partial\psi}{\partial x^n}]$ is positive.

An n form ω on an n dimensional submanifold is a smooth choice of an $\omega(s) \in \det(T_s\Sigma^*)$ for every $s \in \Sigma$. To say what smooth means we choose a parametrisation $\psi\colon U \to \Sigma$. Then if $\psi(x) = s$ we have

$$\omega(s) = \omega_\psi(x)\left[\frac{\partial\psi}{\partial x^1}(x), \dots, \frac{\partial\psi}{\partial x^n}(x)\right]$$

where from Prop. 6.5 we have that

$$\omega_\psi(x) = \omega(\psi(x))\left(\frac{\partial\psi}{\partial x^1}(x), \dots, \frac{\partial\psi}{\partial x^n}(x)\right).$$

We call ω smooth if whenever we choose a parametrisation like this the function $\omega_\psi\colon U \to \mathbb{R}$ is smooth.

Proposition 6.7. *If $\chi\colon V \to \Sigma$ and $\psi\colon U \to \Sigma$ are oriented parametrisations with $\chi = \psi \circ \rho$ for $\rho\colon U \to V$ then*

$$\det(J(\rho)) > 0.$$

Proposition 6.8. *If $\chi\colon V \to \Sigma$ and $\psi\colon U \to \Sigma$ are parametrisations with $\chi = \psi \circ \rho$ for $\rho\colon U \to V$ then*

$$\omega_\chi = \det(J(\rho))\omega_\psi \circ \rho.$$

Note 6.1. This proposition shows that if ω_ψ is smooth then so also is ω_χ.

Definition 6.9. If ω is a smooth n-form on a submanifold Σ we define its support to be the closure of the set of points at which it is not zero.

Definition 6.10. If $\psi\colon U \to \Sigma$ is an oriented parametrisation and ω a smooth n form with support in $\psi(U)$ we define

$$I_\psi(\omega) = \int_U \omega_\psi \circ \psi dx^1 \dots dx^n.$$

Proposition 6.11. *If $\chi\colon V \to \Sigma$ is another oriented parametrisation then $I_\psi(\omega) = I_\chi(\omega)$.*

Note 6.2. To be sure that the integral exists we should really require that the support of ω be compact, ie closed and bounded in $\mathbb{R}^N$.

Definition 6.12. Let $\{W_1, \dots, W_k$ be an open cover of Σ. A partition of unity subordinate to this cover is a collection of smooth functions $\rho_i\colon \Sigma \to [0, \infty) \subseteq \mathbb{R}$ such that $\text{supp}(\rho_i) \subseteq W_i$ and $\sum_i \rho_i s = 1$.

If ω is an n form with support in the image of some (oriented) parametrisation $\psi\colon U \to \Sigma$ then we define

$$\int_\Sigma \omega = I_\psi(\omega).$$

If ω is a more general form we assume that it is possible to find a collection of parametrisations $\psi_i\colon U_i \to \Sigma$ with $\Sigma = \bigcup \psi_i(U_i)$ and a partition of unity subordinate to $\{\psi_i(U_i)\}$ and we define

$$\int_\Sigma \omega = \sum_i \int_{\psi_i} \rho_i \omega.$$

Note 6.3. (1) The support of $\rho_i\omega$ is in $\psi_i(U_i)$.
(2) The existence of the partition of unity required is guaranteed in much generality which we will not go into in this course. We shall assume it exists.

Proposition 6.13. *The integral of an n form just defined is independent of the choice of parametrisations and partition of unity.*

[**2011: End of Lecture 24**]

6.3. **Volume forms.**

Lemma 6.14. *If V is a vector space with an inner product $\langle\ ,\ \rangle$ and $e_1,\dots,e_n$ and $f_1,\dots,f_n$ are orthonormal bases then $[e_1,\dots,e_n] = \pm[f_1,\dots,f_n]$.*

Definition 6.15. If $S \subseteq \mathbb{R}^N$ is an oriented submanifold and $e_1,\dots,e_n$ is an orthonormal basis of T_sS and $[e_1,\dots,e_n]$ is positive we define $\mathrm{vol}_S(s) = [e_1,\dots,e_n]$, the volume form of S.

Note 6.4. (1) Orthonormal means orthonormal with respect to the inner product on $\mathbb{R}^N$ restricted to T_sS.
(2) The Lemma guarantees that the volume form is independent of the choice or orthonormal basis.
(3) When its obvious from context we will drop the S from vol_S and just write vol.

Proposition 6.16. *Let $\psi\colon U \to S$ be a parametrisation of S. Let $g_{ij} = \langle \frac{\partial\psi}{\partial x^i}, \frac{\partial\psi}{\partial x^j}\rangle$ then*

$$\mathrm{vol}_S = \sqrt{\det(g_{ij})}\left[\frac{\partial\psi}{\partial x^1},\dots,\frac{\partial\psi}{\partial x^n}\right].$$

Note 6.5. It follows that vol_S is a smooth n form.

Proposition 6.17. *If $\Sigma \subseteq \mathbb{R}^3$ is an oriented surface and n is the unit normal to Σ at s which defines the orientation then*

$$\mathrm{vol}_\Sigma(v,w) = \langle v \times w, n\rangle$$

for any v and w in $T_s\Sigma$.

Corollary 6.18. *If Σ is an oriented surface then $R\,\mathrm{vol}_\Sigma(X,Y) = \langle \Pi(X)\times\Pi(Y), n\rangle$.*

6.4. **One forms.** If V is a vector space we call a linear map $V \to \mathbb{R}$ a one-form and denote the set of them all by V^*. If $v_1,\dots,v^n$ is a basis of V we can define $v_i^* \in V^*$ by requiring that $v_i^*(v_j) = \delta_{ij}$. The collection $v_1^*,\dots,v_n^*$ is a basis of V^* called the *dual basis* to $v_1,\dots,v_n$.

If V is a two dimensional vector space and α and β are one forms we define $\alpha\wedge\beta$, their *wedge product*, by

$$(\alpha\wedge\beta)(v,w) = \alpha(v)\beta(w) - \alpha(w)\beta(v).$$

for any v and w in V.

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Let S be a submanifold and $f\colon S \to \mathbb{R}$ a smooth function. Then for any $s \in S$ we have that $df(s) = f'(s)\colon T_s\Sigma \to \mathbb{R}$ is a linear map. In particular if $\psi\colon U \to S$ is a parametrisation and $\hat\psi = \psi^{-1}$ then each of the components of $\hat\psi = (\hat\psi^1,\dots,\hat\psi^n)$ is a function $\hat\psi^i\colon \psi(U) \to \mathbb{R}$ and hence $d\hat\psi^i(s)\colon T_s\Sigma \to \mathbb{R}$ is a linear map for any $s \in \psi(U)$. We have

Proposition 6.19. *The linear maps $d\hat\psi^1(s),\dots,d\hat\psi^n(s)$ are a basis of the dual space T_sS^* satisfying*

$$d\hat\psi^i(s)\left(\frac{\partial\psi}{\partial x^j}(x)\right) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

if $\psi(x) = s$.

Definition 6.20. A one form η is a choice of linear map $\eta(s)\colon T_sS \to \mathbb{R}$ for every s. Any one form can be expanded as $\eta(s) = \sum_{i=1}^n \eta_i(s)d\hat\psi^i(s)$ and we say η is smooth if each of the $\eta_i\colon \psi(U) \to \mathbb{R}$ is smooth.

Lemma 6.21. *If $\psi\colon U \to \Sigma$ is a parametrisation of a surface Σ then*

$$d\hat\psi^1 \wedge d\hat\psi^2 = \left[\frac{\partial\psi}{\partial x^1}, \frac{\partial\psi}{\partial x^2}\right].$$

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Proposition 6.22. *(1) If $\psi\colon U \to \Sigma$ is a parametrisation of a surface Σ and η is a one form then the two form $d\eta$ defined by*

$$\begin{aligned} d\eta(s) &= \sum_{i=1}^2 d\eta_i(s)\wedge d\hat\psi^i(s) \\ &= \left(\frac{\partial \eta_2\circ\psi}{\partial x^1}(\psi^{-1}(x)) - \frac{\partial\eta_1\circ\psi}{\partial x^2}(\psi^{-1}(x))\right) d\hat\psi^1(s)\wedge d\hat\psi^2(s). \end{aligned}$$

is independent of the parametrisation.
(2) If f is a function on Σ then $d(f\eta) = df \wedge \eta + f d\eta$.

A closed surface is a two dimensional submanifold of $\mathbb{R}^3$ which is closed and bounded. It follows that it has no boundary.

Proposition 6.23 (Weak Green's Theorem). *If η is a one form on an oriented closed surface Σ then $\int_\Sigma d\eta = 0$.*

7. Gauss-Bonnet Theorem

Proposition 7.1. *Let Σ_t be a family of closed oriented surfaces in $\mathbb{R}^3$ depending smoothly on a parameter t. Define $\eta(X) = \langle \frac{dn}{dt} \times dn(X), n\rangle$ then*

$$\frac{d}{dt} R_t \operatorname{vol}_t = d\eta.$$

Corollary 7.2. *Let Σ_t be a family of closed oriented surfaces in $\mathbb{R}^3$ depending smoothly on a parameter t. Then $\int_{\Sigma_t} R_t \operatorname{vol}$ is independent of t.*

Recall that for a sphere we have

$$\frac{1}{2\pi}\int_{S^2} R \operatorname{vol} = 2.$$

Definition 7.3. We say a surface Σ' is obtained from a surface Σ by *adding a handle* if we remove two disks from Σ and attach to the two resulting circles in Σ each end of a cylinder.

[**2011: End of Lecture 27**]

Proposition 7.4. *If the oriented closed surface Σ' is obtained from the oriented closed surface Σ by adding a handle then*

$$\frac{1}{2\pi}\int_{\Sigma'} R \operatorname{vol} = \frac{1}{2\pi}\int_{\Sigma} R \operatorname{vol} - 2$$

Corollary 7.5. *If Σ is obtained from a sphere by adding g handles then*

$$\frac{1}{2\pi}\int_{\Sigma} R \operatorname{vol} = 2 - 2g.$$

It is a theorem in topology that if Σ is a closed surface in $\mathbb{R}^3$ then Σ is homeomorphic to a sphere with g handles.

7.1. Tessellations.

Definition 7.6. Let Σ be a surface in $\mathbb{R}^3$. A tessellation T for Σ is a decomposition of Σ into vertices, edges and faces such that each face is homeomorphic to a polygon.

A tessellation where every face has three edges is called a triangulation.

Definition 7.7. Let T be a tessellation of a surface Σ and let v be the number of vertices, e the number of edges and f the number of faces. Define

$$\chi(\Sigma, T) = v - e + f.$$

Proposition 7.8. *Let Σ be a surface and T and T' tessellations. Then $\chi(\Sigma, T) = \chi(\Sigma, T')$.*

If Σ is a surface we define its Euler characteristic $\chi(\Sigma)$ to be $\chi(\Sigma, T)$ for some some tessellation.

Proposition 7.9. *If Σ is a surface of genus g, ie it is obtained from a sphere by adding g handles then $\chi(\Sigma) = 2 - 2g$.*

Note 7.1. This gives an alternative way of calculating $g(\Sigma)$ namely $g(\Sigma) = (1/2)(2 - \chi(\Sigma))$.

Theorem 7.10 (Gauss-Bonnet)**.** *If Σ is a closed oriented surface then*

$$\frac{1}{2\pi}\int_{\Sigma} R\,\mathrm{vol} = \chi(\Sigma) = 2 - 2g(\Sigma).$$

[**2011: End of Lecture 28**]