Examination in School of Mathematical Sciences
Semester 2 2011

105919  Geometry of Surfaces III – MOCK EXAM
PURE MTH 3022

Official Reading Time: 10 mins
Writing Time: 120 mins
Total Duration: 130 mins

NUMBER OF QUESTIONS: 5  TOTAL MARKS: 70

Instructions

• Attempt all questions.
• Approximate marks are indicated.
• Begin each answer on a new page.
• Examination materials must not be removed from the examination room.

Materials

• 1 Blue book is provided.
• Calculators are not permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.
1. For each of the following statements say whether it is True or False. Give a very short justification or counter-example in each case.

(a) The curve $\gamma(t) = (t, t^2, t^{99})$ is arc-length parametrised.

(b) A curve with constant unit tangent $T$ has zero curvature $\kappa$.

(c) It is possible to find a tessellation of the torus with 100 vertices, 92 edges and 76 faces.

(d) A cylinder has Gaussian curvature which is everywhere positive.

(e) There exists a surface with principal curvatures $\lambda_1 > 0$ and $\lambda_2 < 0$ and Gaussian curvature zero.

[10 marks]

2. (a) Define the tangent space, $T_s S$ to a submanifold $S \subset \mathbb{R}^N$ at a point $s \in S$.

(b) Assume that $\psi: U \to S$ is a parametrisation of $S$ where $U$ is open in $\mathbb{R}^n$. Show that $T_s S = \text{im}\, \psi'(x)$ if $s = \psi(x)$. You may assume that $\text{dim}(T_s S) = n$.

(c) Consider the submanifold defined by the equation $x^2 + y^2 = z$ in $\mathbb{R}^3$. Find the tangent space at a general point.

[10 marks]

3. (a) Let $\gamma: (a, b) \to \mathbb{R}^3$ be a parametrised curve. Define $T$, $N$ and $B$.

(b) Consider the parametrised curve $C$ with parametrisation $\gamma(t) = (t, t, \frac{t^2}{2})$.

   (i) Is $\gamma$ an arc-length parametrisation of $C$? Justify your answer.

   (ii) Find $T$, $N$ and $B$ for $C$.

   (iii) Compute the curvature $\kappa$ and the torsion $\tau$ of the curve $C$.

     You may assume the Frenet Formulae: $\dot{T} = \kappa N$, $\dot{N} = -\kappa T + \tau B$ and $\dot{B} = -\tau N$.

[20 marks]

4. Consider the paraboloid $P$ defined by the equation $x^2 + y^2 = z$.

   (a) Find a parametrisation whose image is all of $P$ (do not prove it is a parametrisation just write it down).

   (b) Compute the normal to $P$ oriented to that it points up at $(0, 0, 0)$.

   (c) Compute the Mean and Gaussian curvature of $P$.

[15 marks]
5. (a) (i) Define what the volume form $\text{vol}_S$ of an $n$ dimensional oriented submanifold $S$ is.

(ii) If $f : S \rightarrow \mathbb{R}$ is a smooth function explain how you would integrate $f \text{vol}_S$ over the image of an oriented parametrisation $\psi : U \rightarrow S$ where $U \subset \mathbb{R}^n$ is open.

(b) Consider the surface $T$ in $\mathbb{R}^3$ defined by the equation

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2$$

for $0 < r < R$.

(i) Sketch this surface, draw a tessellation on it and compute its Euler characteristic.

(ii) If $R$ is the Gaussian curvature of this surface what does the Gauss Bonnet theorem predict that $\frac{1}{2\pi} \int_T R \text{vol}_T$ is? (Don’t try and compute this integral.)

[15 marks]