2-2 Mean Value Thm

Recall that if \( \phi : (a, b) \to \mathbb{R} \), differentiable on \((a, b)\), & \phi(x) \text{ is continuous on } [a, b] \) & then

\[ f \in (a, b) \text{ s.t.} \]

\[ \phi(b) - \phi(a) = \phi'(f) (b-a) \]  \hspace{1cm} \text{(MaxThm)} \]

Let \( f : U \to \mathbb{R}^m \) differentiable & \( u \in \mathbb{R}^n \).

Let \( x_0, x_1 \in U \) & assume

\[ [x_0, x_1] = \{ t x_1 + (1-t) x_0 / 0 \leq t \leq 1 \} \subseteq U, \]

& consider \( \phi_u : (0, 1) \to \mathbb{R}^n \)

\[ \phi_u(t) = \langle u, f(t x_1 + (1-t) x_0) \rangle \]

Then \( \phi_u(t) \) is differentiable at \( t = 0 \).

(Chain rule), \( \phi_u(0) = \langle u, f(x_0) \rangle \).

Let \( f \in (a, b) \text{ s.t.} \)

\[ \phi(1) - \phi(0) = \phi'(f) \]

But \( \phi(1) = \langle u, f(t x_1) \rangle \)

\[ \phi(0) = \langle u, f(x_0) \rangle \]

\[ \phi'(f) = \frac{d}{dt} \sum_i u_i f^i (t x_1 + (1-t) x_0) \bigg|_{t=1} = \]

\[ \sum_i u_i \frac{df^i}{dx_j} (t x_1 + (1-t) x_0) (x_i - x_0) \]
\[ \langle f'(\xi_n)(x_1-x_0), u \rangle \]

\[ \xi_n = 3x_1 + (1-3)x_0. \]

Hence we proved:

**Theorem 2.18 (Mean Value Theorem)**

Let \( U \subseteq \mathbb{R}^n \) be open, \( f: U \to \mathbb{R}^m \) differentiable on \([x_0, x_1] \subseteq U\), then there exists a \( \xi_n \in (x_0, x_1) \) such that

\[ \langle f(x_1), u \rangle = \langle f(x_0), u \rangle + \langle f'(\xi_n)(x_1-x_0), u \rangle \]

**Example**

\( u = e^i \);

\[ f'(x) = f'(x_0) + f''(\xi_n)(x_1-x_0) \]

Can't expect to get \( f(x) = f(x_0) + \ldots \)

**Corollary 2.19**

If \( U \) is open in \( \mathbb{R}^n \) and \( h: U \to \mathbb{R}^m \) is differentiable with \( \| h'(x) \| \leq \delta \) for all \( x \in [x_0, x_1] \subseteq U \), then

\[ \| h(x_0) - h(x_1) \| \leq \delta \| x_0 - x_1 \| \]

**Proof**

Let \( u = h(x_1) - h(x_0) \), then

\[ \| u \| = \langle h(x_1)-h(x_0), u \rangle \]

\[ = \langle h'(\xi_n)(x_1-x_0), u \rangle \]
\[ \| u \| ^2 \leq \| h'(x_0) \| \| x_1 - x_0 \| \| u \| \]

\[ \| u \| \leq \delta \| x_1 - x_0 \| \]

\[ \| h(x_1) - h(x_0) \| \leq \delta \| x_1 - x_0 \| \]

Recall we defined a norm before \( \| \cdot \| \) as a matrix \( A \)

\[ \| A \| = \sup_{x \in \mathbb{R}^n} \{ \| A(x) \| : \| x \| = 1 \} \]

Then we have \( \| A \| \| x \| \leq \| A \| \| x \| \).

We can also use

\[ \| A \|_2 = \sqrt{\sum_{i,j=1}^{m,n} (A_{ij})^2} \]

As

\[ \| A x \| ^2 = \sum_{i,j=1}^{m,n} |A_{ij}|^2 x_i^2 \]

\[ = \sum_{i,j=1}^{m,n} |\langle A_i, x \rangle|^2 \]

\[ \leq \sum_{i=1}^{m} \| A_i \| ^2 \| x \| ^2 \]

\[ = \sum_{i=1}^{m} \| A_i \| ^2 \| x \| ^2 \]

\[ \leq \left( \sum_{i=1}^{m} |A_{ij}|^2 \right) \| x \| ^2 \]

\[ \| A x \| \leq \| A \|_2 \| x \| \]
\[\| \mathbf{u} \| < c \| \mathbf{h}(\mathbf{u}) \| \| \mathbf{x}_1 - \mathbf{x}_0 \|\]

\[\| \mathbf{u} \| < c \| \mathbf{x}_1 - \mathbf{x}_0 \|\]

\[\| \mathbf{h}(\mathbf{x}_1) - \mathbf{h}(\mathbf{x}_0) \| \leq \varepsilon \| \mathbf{x}_1 - \mathbf{x}_0 \|\]

\[\text{Notice on here all we need is } \| \mathbf{Ax} \| \leq \| \mathbf{A} \| \| \mathbf{x} \| \text{ so}\]

\[\text{The Inverse Function Theorem }\]

Recall that if \( \mathbf{u} \) are variables of \( f(\mathbf{a}) \) to \( \mathbf{e} \)

\[eg f'(\mathbf{a}) \neq 0\]

\[\Sigma \quad 1\]

\[f \in \mathbb{R}^n \rightarrow \mathbb{R}^n] \quad f: U \rightarrow f(U) \text{ is } 1-1\]

& onto (because \( f \) is monotonic) near \( \mathbf{a} \)

\[\therefore \text{ an inverse } f^{-1}: V \rightarrow U. \text{ It turns out that if } \mathbf{f} \text{ is } C^k \text{ so also is } f^{-1} \text{ & } (f^{-1})'(\mathbf{x}) = \frac{m^{-1}}{f''(\mathbf{t})}\]

\[\text{we want a more complicated version:}\]
In the most general case, assume

\[ U \subseteq \mathbb{R}^n \quad \alpha \in U \quad f: U \to \mathbb{R}^n \]

\[ f(U) \subseteq \mathbb{R}^n \quad \text{open in } \mathbb{R}^n \]

Assume \( \exists g : \overline{U} \to U \quad g = f^{-1} \)

Then \( \frac{g \circ f}{g \circ f} = \text{id} \)

\[ \text{id}_x(x) = x \]

Note: \( \text{id}^{-1}(x) = \text{id} \) (linear).

So by chain rule:

\[ f'(g \circ f) \]

\[ g'(f(a)) \circ f'(a) = \text{id} \]

\[ g'(f(a)) = f'(a)^{-1} \]

So at least \( f'(a) \) is invertible as a linear map.
Theorem (Inverse Function Theorem)

Let \( U \subseteq \mathbb{R}^k \) be open and \( f: U \rightarrow \mathbb{R}^k \) be \( C^k \) for \( k \geq 1 \). Assume that \( f'(x) \) is invertible. Then \( \exists V \subseteq \mathbb{R}^n \) such that \( x \in V \subseteq U \) and

1. \( f(V) \) is open in \( \mathbb{R}^n \)
2. \( f: V \rightarrow f(V) \) is invertible
3. \( f': f(V) \rightarrow V \) is \( C^k \)
4. \( (f^{-1})'(f(a)) = [f'(a)]^{-1} \)

Proof: First we simplify. If we define

\[
\tilde{f}(x) = f'(a)^{-1} \left[ f(x + a) - f(a) \right]
\]

then \( \tilde{f}(0) = 0 \) and it satisfies the hypotheses. Thus we assume \( f(0) = 0 \) and

\[
\|f'(0)\|_2 = \sup \{ \|f'(0)(x)\|/\|x\| = 1 \} = 1
\]

and

\[
\|f'(0)\|_2 = \sup \{ \|x\|/\|x\| = 1 \} = 1
\]
Claim: If \( x \in \text{B}(0, r) \), then \( \|x\|_2 < r \). 

Proof: Let \( x = (x_1, x_2, \ldots, x_n) \). Then, we want to show that \( \|x\|_2 < r \).

By the Cauchy-Schwarz inequality,
\[
\|x\|_2^2 = \sum_{i=1}^{n} x_i^2 < r^2.
\]

Thus, \( \|x\|_2 < r \), as desired.