§ 2-2 Mean Value Th^m (Lecture 9) ①

Recall that if $\phi : [a, b] \to \mathbb{R}$, differentiable on (a, b), & cts on $[a, b]$ & then

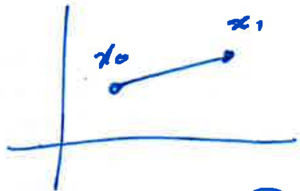
$\exists\, \xi \in (a, b)$ s.t.

$$\phi(b) - \phi(a) = \phi'(\xi)(b-a)$$

(Maths 1)

Let $f : \underset{\substack{\text{open} \\ \mathbb{R}^n}}{U} \to \mathbb{R}^m$ diff'ble and $u \in \mathbb{R}^n$.

Let $x_0, x_1 \in U$ & assume



$$[x_0, x_1] \overset{\text{def}}{=} \{ t x_1 + (1-t) x_0 \,/\, 0 \le t \le 1 \} \subseteq U.$$

& $\underset{\text{}}{\text{for } u \in \mathbb{R}^m}$ consider $\phi_u(t) = \langle u, f(t x_1 + (1-t) x_0 \rangle$

Then $\phi_u(t)$ is diff'ble on $(0, 1)$ (chain rule), cts on $[0, 1]$. $\therefore$

$\exists\, \xi$ s.t

$$\phi(1) - \phi(0) = \phi'(\xi)$$

But

$$\phi(1) = \langle u, f(x_1) \rangle$$

$$\phi(0) = \langle u, f(x_0) \rangle$$

$$\phi'(\xi) = \frac{d}{dt} \sum u^i f^i (t x_1 + (1-t) x_0) \Big|_{t=\xi}$$

$$= \sum u^i \frac{\partial f^i}{\partial x^j} (\xi x_1 + (1-\xi) x_0) \, (x_1 - x_0)^j$$

$$= \langle f'(\xi_u)(x_1 - x_0), u\rangle$$

$$\xi_u = \xi x_1 + (1-\xi)x_0.$$

Hence we proved:

Theorem 2.18 (Mean Value Theorem)

~~Let~~ $\underset{\text{open}}{U \subseteq} \mathbb{R}^n$, $f : U \to \mathbb{R}^m$ diff'ble &

$[x_0, x_1] \subseteq U$, then ~~$\exists \xi_u \in$~~ $\forall u \in \mathbb{R}^m$

$\exists \xi_u \in [x_0, x_1]$ s.t.

$$\langle f(x_1), u\rangle = \langle f(x_0), u\rangle + \langle f'(\xi_u)(x_1 - x_0), u\rangle$$

Example $u = e^i$ $\quad f^i(x) = f^i(x_0) + f^{i\prime}(\xi_{e^i})(x_1 - x_0)$

~~x_0~~ ~~.....~~ Can't expect to get $f(x) = f(x_0) + \ldots$

Corollary 2.19 If U is open in $\mathbb{R}^n$ & $h : U \to \mathbb{R}^m$

is diff'ble with $\|h'(\xi)\| \leq \varepsilon$ $\forall \xi \in [x_0, x_1] \subseteq U$

then $\|h(x_0) - h(x_1)\| \leq \varepsilon \|x_0 - x_1\|$

Proof Let $u = h(x_1) - h(x_0)$ then

$$\|u\| = \langle h(x_1) - h(x_0), u\rangle$$

$$= \langle h'(\xi_u)(x_1 - x_0), u\rangle$$

$$\therefore \|u\|^2 \le \|h'(\xi_u)\| \, \|x_1 - x_0\| \, \|u\|$$

$$\therefore \|u\| \le \varepsilon \|x_1 - x_0\|$$

$$\|h(x_1) - h(x_0)\| \le \varepsilon \|x_1 - x_0\| \quad //$$

Recall we defined once before $\|\cdot\|$ for a matrix A

$$\|A\| = \sup_{x \in \mathbb{R}^n} \{ \|A(x)\| \mid \|x\| = 1 \}$$

This satisfies $\|Ax\| \le \|A\| \, \|x\|$.

We could also use

$$\|A\|_2 = \sqrt{\sum_{i,j=1}^{n,m} |A_{ij}|^2}$$

as

$$\|Ax\|^2 = \sum_{i=1}^{m} \Big| \sum_{j=1}^{n} A_{ij} x^j \Big|^2$$

Row

$$= \sum |\langle A_i, x \rangle|^2$$

$(A_{i1} \ldots A_{in})$

$$A_i = \begin{pmatrix} A_{i1} \\ \vdots \\ A_{in} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\le \sum \|A_i\|^2 \, \|x\|^2$$

$$= (A_{i1})^2 + \ldots + (A_{in})^2 \, \|x\|^2$$

$$\therefore \le \underbrace{\Big(\sum |A_{ij}|^2 \Big)}_{\|A\|_2} \|x\|^2 \qquad \therefore \|Ax\| \le \|A\|_2 \, \|x\|$$

$\therefore \quad \|u\|^2 \leq \|h'(\xi_n)\| \, \|x_1 - x_0\| \, \|u\|$

$\therefore \quad \|u\| \leq \varepsilon \|x_1 - x_0\|$

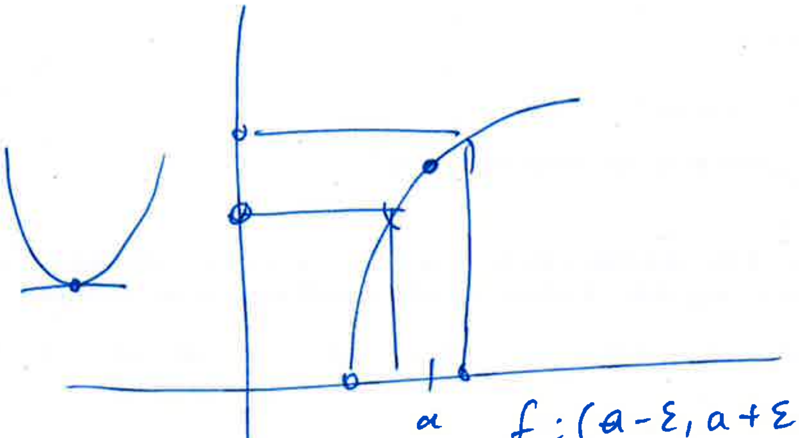
$\therefore \quad \|h(x_1) - h(x_0)\| \leq \varepsilon \|x_1 - x_0\|$

~~Notice on here all we need is~~ ~~$\|Ax\| \leq \|A\|\|x\|$~~ so

§2.3 ~~...~~ The Inverse Functn Thm

Recall that in one variable if $f'(a) \neq 0$

eg $f'(a) > 0$



$f : (a-\varepsilon, a+\varepsilon) \to (f(a-\varepsilon), f(a+\varepsilon))$

$\exists$ $\varepsilon > 0$ ~~$U \ni a$~~ ~~... s.t~~ ~~$f : U \to f(U)$~~ is 1-1

& onto (because f is monotonic ~~increasing~~) near a)

$\therefore$ $\exists$ an inverse $f^{-1} : V \to U$. It turns out that if f is C^k so also is f^{-1} & $(f^{-1})'(t) = \frac{1}{f'(t)}$

We want a more complicated version:

In the more general case assume

$\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n$ $\quad a \in \mathcal{U}$ $\quad f : \mathcal{U} \to \mathbb{R}^m$

$f(\mathcal{U})$ ~~...~~ open in $\mathbb{R}^n$

Assume $\exists\, g : \overset{f(\mathcal{U})}{\cancel{\mathbb{R}}} \to \mathcal{U}$ $\quad g = f^{-1}$

Then $\overset{g \circ f}{\cancel{f \circ g}} = id_{\cancel{\mathbb{R}}}$ $\qquad id_{\cancel{\mathbb{R}}}(x) = x$

Note : $\quad id'(x) = id \quad$ (linear).

So by chain rule :

$\cancel{f'(gt}$

$$g'(f(a)) \circ f'(a) = id$$

$$\therefore \quad g'(f(a)) = f'(a)^{-1}$$

So at the very least need $f'(a)$ invertible as a linear map.

Theorem (Inverse Function Theorem)

Let $U \subseteq \mathbb{R}^{n}$ be open and $f : U \to \mathbb{R}^{n}$ be C^k for $k \geq 1$. Assume that $f'(a)$ is invertible. Then $\exists V \underset{\text{open}}{\subseteq} \mathbb{R}^n$ such that $a \in V \subseteq U$ &

1. $f(V)$ is open in $\mathbb{R}^n$
2. $f : V \to f(V)$ is invertible
3. $f^{-1} : f(V) \to V$ is C^k
4. $(f^{-1})'(f(a)) = [f'(a)]^{-1}$

Proof First we simplify. If we

$$\tilde{f}(x) = f'(a)^{-1}\left[f(x+a) - f(a)\right]$$

then $\tilde{f}(0) = 0$ & $\tilde{f}(\cancel{\phi}\,0) = \cancel{\mathbb{I}}\, I$

So WLOG let $a = 0$ and assume $f(0) = 0$

& $f'(0) = \cancel{\mathbb{I}}.\mathrm{id}$ ~~So~~

$$\cancel{\|f'(0)\|_2 = \sup\{\|f'(0)(x)\| \mid \|x\| = 1\}}$$

$$\cancel{= \sup\{\|x\| \mid \|x\| = 1\} = 1}$$

as the partial derivatives all depend continuously on x we have that $\|I-f'(x)\|_2^2$ and hence $\|I-f'(x)\|_2$ is cts

$\therefore$ can find r s.t. if $\|x\|<r$

then

$$\|I-f'(x)\|_2 \le \frac{1}{2} \qquad \left(\|I-f'(0)\| = 0\right)$$

$$\|0\|$$

End here

<u>Claim</u> If $x_0, x_1 \in \overline{B(0,r)}$ then

$$\|(x_1-x_0)-(f(x_1)-f(x_0))\| < \frac{1}{2}\|x_1-x_0\|$$

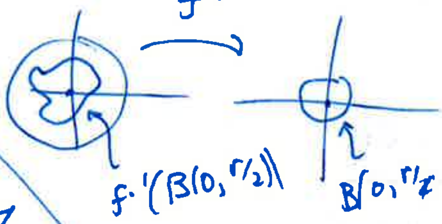
<u>Proof</u> Let ~~the~~ $h(x) = x - f(x)$ & use Corollary to MVT as

$$h'(x)(h) = (I-f'(x))(h) \quad //$$

We want to show that if $\|y\| < r/2$

then $\exists!\, x \in B(0,r)$ s.t. $f(x) = y$

Let $g(x) = x + y - f(x)$ then



$$g(x) = x \iff f(x) = y$$

Want to use CMTh : $g : \overline{B(0,r)} \to \overline{B(0,r)}$