Lecture 8

Let's change the def'n of partial.
Let's make it:

\[ f : U \to \mathbb{R}^m \]

\[ \nabla f \]

\[ \mathbb{R}^n \]

\[ \frac{df^i}{dx_j}(a) = \frac{d}{dt} \left( f^i(a + te^j) \right) \]

\[ = \lim_{t \to 0} \frac{f^i(a + te^j) - f^i(a)}{t} \]

Then Corollary 2.12 makes more sense and proves that:

\[ J(f)(a) = \begin{bmatrix} \frac{df^i}{dx_j} \\ \end{bmatrix}_{(ij)} \]

\[ : \mathbb{R}^n \to \mathbb{R}^m \]

\[ \sum_{i=1}^{m} \sum_{j=1}^{n} \]
Prop. 2.15

If $f \in C^1(U)$ then $f$ is differentiable at all $a \in U$.

Proof

A little messy to write out so we set $U \subseteq \mathbb{R}^2$ and $a = (0,0) \in U$.

We expect $f'(a) = \left( \frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial x^2}(0,0) \right)$.

Consider

\[ \| f(x + h) - f(0) - \frac{\partial f}{\partial x}(0,0) h^1 - \frac{\partial f}{\partial x^2}(0,0) h^2 \| \leq \cdots \]

\[ = \| f(h^1, h^2) - f(h^1, 0) + f(h^1, 0) - f(0,0) - \frac{\partial f}{\partial x}(0,0) h^1 - \frac{\partial f}{\partial x^2}(0,0) h^2 \| + \frac{\partial f}{\partial x^2}(h^1,0) h^2 \| \leq \]

\[ \leq \| f(h^1, h^2) - f(h^1, 0) - \frac{\partial f}{\partial x}(h^1,0) h^2 \| - A \]

\[ + \| f(h^1, 0) - f(0,0) - \frac{\partial f}{\partial x}(0,0) h^1 \| - B \]

\[ + \| \frac{\partial f}{\partial x^2}(h^1,0) - \frac{\partial f}{\partial x^2}(0,0) \| \cdot |h^2| \]

\[ \leq \frac{A}{|h^2|^2} + \frac{B}{|h^2|^2} + \frac{C|h^2|}{|h^2|^2} \]

\[ \rightarrow 0 \text{ as } \| h \| \rightarrow 0. \]
Prop 2.16

If \( f \in C^2(U, \mathbb{R}) \), then \( \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} \).

Similarly, if \( f \in C^k \) then all partial derivatives of order up to and including \( k \) are independent of the order.

Proof

Again, this can be messy, so we prove for \( U \subseteq \mathbb{R}^2 \), \( f \in C^2(U, \mathbb{R}) \), open.

We let \( a = (0, 0) \).

For small enough \( u \), consider

\[
A(u) = \frac{1}{u^2} \left[ f(0, u) - f(0, 0) - f(u, 0) + f(0, 0) \right]
\]

Fix \( y_0 \) and consider \( g(x) = f(x, u) - f(x, 0) \).

\( g \) is \( C^2 \), and

\[
A(u) = \frac{1}{u^2} \left[ g(u) - g(0) \right]
\]

Apply MVT to \( g \).
\[
\frac{g(u) - g(0)}{u} = g'(\xi)
\]

\[
A(u) = \frac{g'(\xi)}{u} = \frac{1}{u} \left[ \frac{\partial^2 f}{\partial x^1 \partial x^1} (\xi, u) - \frac{\partial f}{\partial x^1} (\xi, 0) \right]
\]

Apply MVT again to \( h(x) = \frac{\partial f}{\partial x^1} (\xi, x) - \frac{\partial f}{\partial x^1} (\xi, 0) \)

This is \( C^1 \) & \( \frac{\partial h}{\partial x^2} (x) = \frac{\partial^2 f}{\partial x^2 \partial x^1} (\xi, x) \)

\[
J \eta \ s.t. \quad \frac{h(u) - h(0)}{u} = h'(\eta) = \frac{\partial^2 f}{\partial x^2 \partial x^1} (\xi, \eta)
\]

\[
A(u) = \frac{\partial^2 f}{\partial x^2 \partial x^1} (\xi, \eta)
\]

Repeat swapping (12)

\[
A(u) = \frac{\partial^2 f}{\partial x^1 \partial x^2} (\xi^*, \eta^*)
\]

Now by continuity

\[
\| (x, y) \| < \delta_1, \ s.t. \quad \frac{\partial^2 f}{\partial x^1 \partial x^2} (x, y) - \frac{\partial f}{\partial x^1 \partial x^2} (0, 0) < \frac{\epsilon}{2}
\]

& \( \delta_2 < \eta \quad \| (x, y) \| < \delta_2 \)
\[ \left| \frac{d^2 f}{d x^2 \partial x} (x_1, y_1) - \frac{d^2 f}{d x \partial x^2} (0, 0) \right| < \frac{\varepsilon}{2} \]

Choose \( u \) so that \( \Omega < \frac{\varepsilon}{2} \)

& find \( \varepsilon, \eta \)

\( \varepsilon^*, \eta^* \) as above

\[
= \left| \frac{d^2 f}{d x^2 \partial x} (0, 0) - \frac{d^2 f}{d x \partial x^2} (0, 0) - \frac{d^2 f}{d x^2 \partial x^2} (\xi, \eta) \right| \\
+ \left| \frac{d^2 f}{d x^2 \partial x} (0, 0) - \frac{d^2 f}{d x^2 \partial x^2} (\xi^*, \eta^*) \right| \\
+ \left| \frac{d^2 f}{d x^2 \partial x} (0, 0) - \frac{d^2 f}{d x^2 \partial x} (\xi, \eta) \right| < \varepsilon
\]

\[ \Rightarrow \text{ must be equal} \]
We will generally work with smooth \((C^\infty)\) functions. It would be tempting to do this from the outset but the Inverse Function Theorem needs the notion of differentiability.

Prop \(2.14\)

\[ C^k(U, \mathbb{R}^m) \]

is a vector space under addition and scalar multiplication of functions.

Proof (Ex)

Prop \(2.18\) (Chain rule)

Assume \( U \subseteq \mathbb{R}^n \), \( V \subseteq \mathbb{R}^m \)

\[ f : U \rightarrow \mathbb{R}^m \quad f(u) \in V \\ g : V \rightarrow \mathbb{R}^k \]

\( a \in U \)

If \( f \in C^k(U, \mathbb{R}^m) \) & \( g \in C^k(V, \mathbb{R}^k) \) for \( k \geq 1 \) then \( g \circ f \in C^k(U, \mathbb{R}^m) \)

\[
\frac{d(g \circ f)^i}{dx_j}(a) = \sum_{l=1}^{m} \frac{dg}{dx_l}(f(a)) \frac{df^l}{dx_j}(a)
\]
We do an induction on k.

Case 0: Note that if \( f, g \) are \( C^1 \) so differentiable and \( h, \phi \) are \( C^k \), then \( f \circ g \) is \( C^0 \).

Case 0: If \( f, g \) are \( C^1 \), then

\[
\begin{align*}
\alpha & \rightarrow \frac{df}{dx}(\alpha) \\
\beta & \rightarrow \frac{dg}{dx}(\beta)
\end{align*}
\]

are \( C^0 \).

\[
\therefore \alpha \rightarrow \frac{dg}{dx}(f(\alpha)) \text{ is } C^0
\]

(composition of \( C^k \) is \( C^k \))

\[
\therefore \text{ LHS is continuous in } \alpha.
\]

\( g \circ f \) is \( C^1 \).

The general case: The for \( 1, \ldots, k-1 \), \( f, g \) \( C^k \)

\[
\begin{align*}
\alpha & \rightarrow \frac{df}{dx}(\alpha) \\
\beta & \rightarrow \frac{dg}{dx}(\beta)
\end{align*}
\]

is \( C^{k-1} \).
But by 171 a $f$ is $C^k$ & hence $C^{k-1}$

\[ \frac{dg}{dx} \text{ of } f \] is $C^{k-1}$

\[ \frac{d(gof)}{x} \text{ is } C^{k-1} \]

\[ \text{gof is } C^k \]

//

End full line 8