Lecture 8

Let's change the defn of partial

Let's make it

$f : U \to \mathbb{R}^m$

U open $\subseteq \mathbb{R}^n$

$$\frac{\partial f^i}{\partial x^j}(a) = \frac{d}{dt}\left(f^i(a+te^j)\right)$$

$$= \lim_{t \to 0} \frac{f^i(a+te^j) - f^i(a)}{t}.$$

Then Corollary 2.12 makes more sense and proves that

$$J(f)(a) = \left[\frac{\partial f^i}{\partial x^j}\right]_{(i,j)} : \mathbb{R}^n \to \mathbb{R}^m$$

$i = 1 \ldots m$

$j = 1 \ldots n$

Prop^n 2.15

If $f \in C'(\mathcal{U})$ then f is diff'ble at all $a \in \mathcal{U}$

Proof

A little messy to write out so we do let $\mathcal{U} \subseteq \mathbb{R}^2$ and $a = (0,0) \in \mathcal{U}$.

We expect $f'(0,0) = \left(\frac{\partial f}{\partial x^1}(0,0), \frac{\partial f}{\partial x^2}(0,0) \right)$

Consider

$$\left\| f(h) - f(0) - \frac{\partial f}{\partial x^1}(0,0) h^1 - \frac{\partial f}{\partial x^2}(0,0) h^2 \right\| \longleftarrow (*)$$

$$= \left\| f(h^1,h^2) - f(h^1,0) + f(h^1,0) - f(0,0) - \frac{\partial f}{\partial x^1}(0,0)h^1 - \frac{\partial f}{\partial x^2}(0,0)h^2 - \frac{\partial f}{\partial x^2}(h^1,0)h^2 + \frac{\partial f}{\partial x^2}(h^1,0)h^2 \right\|$$

$$\leq \left\| f(h^1,h^2) - f(h^1,0) - \frac{\partial f}{\partial x^2}(h^1,0)h^2 \right\| - (A)$$

$$+ \left\| f(h^1,0) - f(0,0) - \frac{\partial f}{\partial x^1}(0,0)h^1 \right\| - (B)$$

$$+ \underbrace{\left\| \frac{\partial f}{\partial x^2}(h^1,0) - \frac{\partial f}{\partial x^2}(0,0) \right\|}_{(C)} |h^2|$$

second partial cts at $(0,0)$!

$$\frac{(*)}{\|h\|} \leq \frac{A}{|h^2|} + \frac{B}{|h^1|} + \frac{C|h^2|}{|h|^2}$$

$\to 0$ as $\|h\| \to 0$. //

Prop^n 2.16

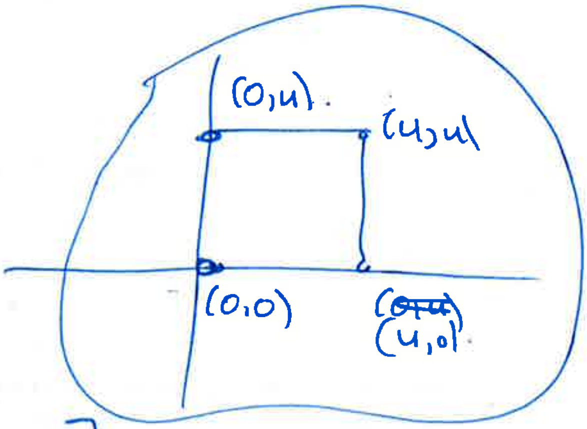
If f ~~is~~ $\in C^2(U, \mathbb{R}^m)$ then $\dfrac{\partial^2 f^l}{\partial x^i \partial x^j} = \dfrac{\partial^2 f^l}{\partial x^j \partial x^i}$

Similarly if $f \in C^k$ then all partial of ~~order~~ degree up to and including k are independent of the order

Proof Again this ~~is~~ can be messy so we ~~[illegible]~~ prove for ~~[illegible]~~ ~~[illegible]~~ $U \subseteq \mathbb{R}^2$ open, $f \in C^2(U, \mathbb{R})$

We let $a = (0,0)$.

For small enough u consider



$$A(u) = \frac{1}{u^2}\Big[f(u,u) - f(0,u) - f(u,0) + f(0,0) \Big]$$

Fix y_0 and consider $g(x) = f(x,u) - f(x,0)$

g is C^2, and

$$A(u) = \frac{1}{u^2}\big[g(u) - g(0)\big]$$

Apply MVTh to g. $\exists\, \xi \in (0,u)$ s.t

$$\frac{g(u) - g(0)}{u} = g'(\xi)$$

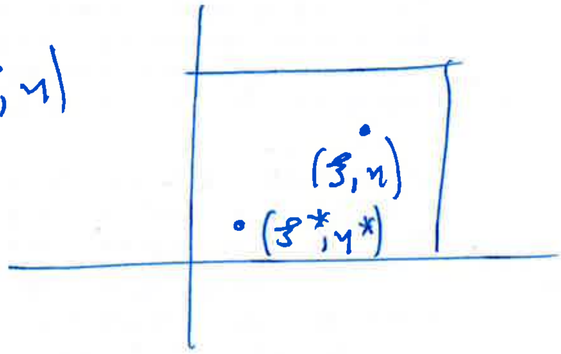
$$\therefore \quad A(u) = \frac{g'(\xi)}{u} = \frac{1}{u}\left[\frac{\partial f}{\partial x^1}(\xi, u) - \frac{\partial f}{\partial x^1}(\xi, 0) \right]$$

Apply MVT again to $h(x) = \frac{\partial f}{\partial x^1}(\xi, x) - \frac{\partial f}{\partial x^1}(\xi, 0)$

This is C^1 & $\frac{\partial h}{\partial x^2}(x) = \frac{\partial^2 f}{\partial x^2 \partial x^1}(\xi, x)$

$\exists\, \eta$ s.t. $\frac{h(u) - h(0)}{u} = h'(\eta) = \frac{\partial^2 f}{\partial x^2 \partial x^1}(\xi, \eta)$

$$\therefore \quad A(u) = \frac{\partial^2 f}{\partial x^2 \partial x^1}(\xi, \eta)$$



Repeat swapping $(1,2)$

$$A(u) = \frac{\partial^2 f}{\partial x^1 \partial x^2}(\xi^*, \eta^*)$$

Now by cty $\exists\, \delta_1$ s.t. $\forall$

$\|(x,y)\| < \delta_1$

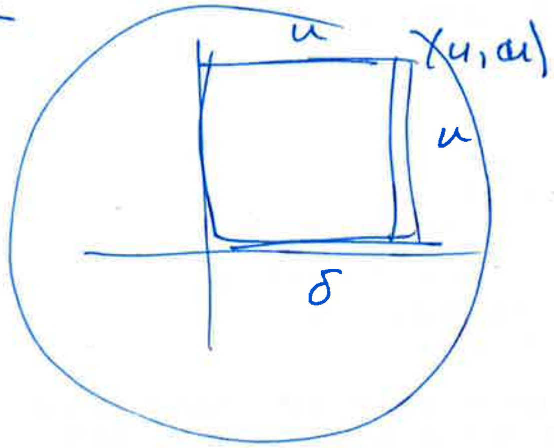
$$\left| \frac{\partial^2 f}{\partial x^1 \partial x^2}(x,y) - \frac{\partial^2 f}{\partial x^1 \partial x^2}(0,0) \right| < \varepsilon/2$$

& δ_2 s.t. $\forall$ $\|(x,y)\| < \delta_2$

$$\left| \frac{\partial^2 f}{\partial x^2 \partial x^k}(x,y) - \frac{\partial^2 f}{\partial x^2 \partial x}(0,0) \right| < \varepsilon/2$$

$\therefore$ $\forall$ ~~to~~ $\|(x,y)\| < \delta = \min\{\delta_1, \delta_2\}$

Choose u ~~so that~~ $< \frac{\delta}{\sqrt{2}}$



& find ξ, η

ξ^*, η^* as above

The $\left| \frac{\partial^2 f}{\partial x^1 \partial x^2}(0,0) - \frac{\partial^2 f}{\partial x^2 \partial x^1}(0,0) \right|$

$$= \left| \frac{\partial^2 f}{\partial x^1 \partial x^2}(0,0) - \frac{\partial^2 f}{\partial x^2 \partial x^1}(0,0) - \frac{\partial^2 f}{\partial x^2 \partial x^1}(\xi,\eta) + \frac{\partial^2 f}{\partial x^1 \partial x^2}(\xi^*,\eta^*) \right|$$

$$\leq \left| \frac{\partial^2 f}{\partial x^1 \partial x^2}(0,0) - \frac{\partial^2 f}{\partial x^1 \partial x^2}(\xi^*,\eta^*) \right| + \left| \frac{\partial^2 f}{\partial x^2 \partial x^1}(0,0) - \frac{\partial^2 f}{\partial x^2 \partial x^1}(\xi,\eta) \right|$$

$< \varepsilon$

$\therefore$ must be equal //

We will generally work with smooth (C^∞) functions. It would be tempting to do this from the outset but ~~some~~ the Inverse Function Theorem needs the notion of differentiability

<u>Prop^n 2.17</u> $C^k(U, \mathbb{R}^m)$ is a vector space under addition and scalar multiplication of functions.

<u>Proof</u> (Ex)

<u>Prop^n 2.18</u> (Chain rule)

Assume $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$, $V \underset{\text{open}}{\subseteq} \mathbb{R}^m$

$f : U \to \mathbb{R}^m$ $\quad f(U) \geq V \quad g : V \to \mathbb{R}^k$

$a \in U$,

If $f \in C^k(U, \mathbb{R}^m)$ & $g \in C^k(V, \mathbb{R}^k)$ for $k \geq 1$

then $g \circ f \in C^k(U, \mathbb{R}^m)$ &

$$\frac{\partial (g \circ f)^i}{\partial x^j}(a) = \sum_{l=1}^{m} \frac{\partial g^i}{\partial x^l}(f(a)) \frac{\partial f^l}{\partial x^j}(a) \qquad (*)$$

Proof

We do an induction on k:

~~Case ①~~ ~~k~~ Note first that f, g are C^1 so differentiable & hence $f \circ g$ diff'ble & ($\ast$) applies.

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Case ① f, g C^1 then

$$\left.\begin{array}{l} a \longmapsto \dfrac{\partial f^{\ell}}{\partial x^{j}}(a) \\[2ex] b \longmapsto \dfrac{\partial g^{i}}{\partial x^{\ell}}(b) \end{array}\right\} \text{ \sout{is C^0} are } C^0$$

$$\therefore \quad a \longrightarrow \frac{\partial g^{i}}{\partial x^{\ell}}(f(a)) \quad \text{is } C^0$$

(composition of cts is cts)

$\therefore$ LHS is continuous in a.

$\therefore$ $g \circ f$ is C^1.

General case: True for $1, 2 \ldots k-1 > 1$, f, g C^k

$$\text{Then} \quad \left.\begin{array}{l} a \longmapsto \dfrac{\partial f^{\ell}}{\partial x^{j}}(a) \\[2ex] b \longmapsto \dfrac{\partial g^{i}}{\partial x^{j}}(b) \end{array}\right\} \text{ is } C^{k-1}.$$

But by 171 a f is C^k & hence C^{k-1}

$$\frac{\partial g^i}{\partial x^l} \circ f \text{ is } C^{k-1}$$

$$\therefore \quad \frac{\partial (g \circ f)^i}{\partial x^j} \text{ is } C^{k-1}$$

$$\therefore \quad g \circ f \text{ is } C^k.$$

//

End Lecture 8
