Recall Chain rule

\[ U \subseteq \mathbb{R}^n \quad \text{open} \quad V \subseteq \mathbb{R}^m \quad \text{open} \]

\[ f : U \rightarrow \mathbb{R}^m \quad f(u) \subseteq V \]

\[ g : V \rightarrow \mathbb{R}^k \]

\[ g \circ f : U \rightarrow \mathbb{R}^k \]

\( f \) differentiable at \( a \) \quad \( g \) differentiable at \( f(a) \)

\[ (g \circ f)'(a) = g'(f(a)) \cdot f'(a). \]

Two special cases of differentiability

1. \( f : (a-\delta, a+\delta) \rightarrow \mathbb{R} \),
   we usually define \( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \).

2. \( \lim_{h \to 0} \frac{|f(a+h) - f(a) - f'(a)h|}{|h|} = 0 \).

The linear function \( f'(a) : \mathbb{R} \rightarrow \mathbb{R} \) is

\( \times \) by the number \( f'(a) \).

Example: Every linear function \( L : \mathbb{R} \rightarrow \mathbb{R} \)

is of the form

\[ L(t) = \alpha t \quad \text{for all} \quad t \]

for some \( \alpha \in \mathbb{R} \).
\( f : (a-\varepsilon, a+\varepsilon) \rightarrow \mathbb{R}^m \)

Can also define
\[
2 f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \in \mathbb{R}^m
\]

Again get:
\[
\lim_{h \to 0} \frac{\|f(a+h) - f(a) - f'(a)\|}{h} = 0
\]

Again the vector \( f'(a) \) is the linear function
\[
\mathbb{R} \rightarrow \mathbb{R}^m
\]
\[
h \mapsto h \cdot f'(a).
\]

Again \( \forall x \): Every linear function \( L : \mathbb{R} \rightarrow \mathbb{R}^m \)
has the form \( L(t) = t \cdot v \) for some \( v \in \mathbb{R}^m \).

Example
In particular, if \( \mathbf{a} \in \mathbb{R}^m \) & \( v \in \mathbb{R}^m \) &
\[
\phi(t) = \mathbf{a} + tv
\]
Then \( \phi'(\mathbf{a}) = v \cdot \mathbf{v} \)
Prop 2.11 \quad f: U \to \mathbb{R}^m \quad \& \quad v \in \mathbb{R}^n

\[ f'(a)(v) = \lim_{t \to 0} \frac{d}{dt} (f(a + tv)) \bigg|_{t=0} \]

Proof \quad \text{let} \quad \phi(t) = a + tv \quad \phi(0) = a

From Prop 2.5 \quad \phi'(0) = v

From Chain rule \quad f(\phi(t)) = (f \circ \phi)(t) = f(a + tv)

is differentiable at 0 \& f is differentiable at a

\[ \frac{d}{dt} (f(a + tv)) \bigg|_{t=0} = f'(a)(\phi'(0)) = f'(a)(v), \]

Corollary 2.12 \quad Let f be as above

and let \quad f = (f_1, \ldots, f_m). \quad \text{let}

\[ e^i = (0, \ldots, 1, \ldots, 0) \quad \text{then} \]

\[ f'(a)(e^i) = \left( \frac{\partial f_1}{\partial x^1}(a), \ldots, \frac{\partial f_m}{\partial x^m}(a) \right) \]
\[ \begin{align*}
\text{def } f'(a)(v) &= \begin{bmatrix}
\frac{df'}{dx_1(a)} & \frac{df'}{dx_2(a)} \\
\vdots & \vdots \\
\frac{df_m}{dx_1(a)} & \frac{df_m}{dx_n(a)}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
\vdots \\
v_n
\end{bmatrix} \\
&= J(f)(a)
\end{align*} \]

where \( \frac{d}{dx} f(a + t e_i) \rvert_{t=0} = \frac{d}{dt} \left( f(a + t e_i) \right) \rvert_{t=0} \)

**Proof** From Prop 2.3, we know that

\[ \frac{d}{dt} \left( f(a + t e_i) \right) \rvert_{t=0} = \left( \frac{df'}{dt} (a + t e_i) \right) \rvert_{t=0} \]

But by def

\[ f'(a)(e_i) = \frac{df}{dt}(a + t e_i) \rvert_{t=0} \]

Then \( \text{def } f'(a)(v) = f'(a) \left( \sum_{i=1}^{n} v_i e_i \right) \)

\[ = \sum_{i=1}^{n} v_i \left( \frac{df'}{dx_1(a)}, \frac{df'}{dx_2(a)}, \ldots, \frac{df_m}{dx_n(a)} \right) \]

\[ = \left[ J(f) \right] \begin{bmatrix}
v_1 \\
\vdots \\
v_n
\end{bmatrix} \text{ as req'd} \]
Note that we have shown that if \( f \) is differentiable at all the partial derivatives of \( f \) exist at \( a \).

2.1 Functions of class \( C^k \)

**Def 2.13** Let \( U \subseteq \mathbb{R}^n \) be open and \( f : U \rightarrow \mathbb{R}^n \). We say that \( f \) is of class \( C^k \) on \( U \) if all partial derivatives of \( f \) on \( U \) up to order \( k \) exist and are continuous.

We write \( C^0 \) for continuous functions and \( C^\infty \) for so-called smooth functions, i.e., that are of class \( C^k \) for all \( k \).

We denote by \( \mathcal{C}^k(U) \) the set of all functions of class \( \mathcal{C}^k \) on \( U \).

**E.G.** \( f \in \mathcal{C}^2(\mathbb{R}^2) \) if \( f \), \( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2^2}, \frac{\partial^2 f}{\partial x_1^2 \partial y} \)
all exist and are cts.

(7.6)

2. Any polynomial is in \( C^\infty(\mathbb{R}) \).
   As are \( \sin(x), \cos(x), e^x \), etc.

3. \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)

\[
f(x_1, x_2) = 7x_1^2 + \sin(x_2)\]

is in \( C^\infty(\mathbb{R}^2) \)

If \( f : U \rightarrow \mathbb{R}^m \) we say \( f \) is \( C^k \) on \( U \) if

\( f^k = (f^1, \ldots, f^m) \) and each \( f^i \) is \( C^k \).

We write \( f \in C^k(U, \mathbb{R}^m) \).

**Lemma 2.19** Let \( a < b < c \) &

1. \( f : (a, c) \rightarrow \mathbb{R} \) cts

2. \( \frac{df}{dx} : f : (a, b) \cup (b, c) \rightarrow \mathbb{R} \) is cts on \( (a, b) \cup (b, c) \)

\[
\lim_{x \to b^+} \frac{df}{dx} = \lim_{x \to b^-} \frac{df}{dx}
\]

Then \( f \) is \( C^1 \) on \((a, c)\).
Proof
Consider MVT on \([b, c]\)

\[
\frac{f(c+b) - f(b)}{n} = f'(c) \quad b < c < t
\]

As \(t \to 0\), \(x \to \begin{array}{c}
\text{Same from below:} \\
\lim_{t \to 0} \frac{f(c+t) - f(b)}{t} = \lim_{t \to 0} f'(c) \\
\lim_{t \to 0} f'(c) = \lim_{t \to 0} f'(b) = \lim_{t \to 0} f'(c)
\end{array}
\]

:: f differentiable \(a, b\) \(f' \) continuous \((a, c)\) \[
\]
Example
Let \(f_p(x) = \begin{cases} x^p & x > 0, \quad p > 0 \\
0 & x \leq 0 \end{cases}\)

\[
f_p^{(k)}(x) = \begin{cases} p(p-1) \cdots (p-k+1) x^{p-k} & x > 0 \\
0 & x \leq 0 \end{cases}\]

If \(q < p\) then \(f \in C^q\) by lemma

If \(q > p\) then \(f \notin C^q\) (Check p?)

\[
f_p(x) = \begin{cases} p^p x^p & x > 0 \\
0 & x \leq 0 \end{cases}
\]
we have seen that if \( f \) is differentiable at a, its partials are defined at \( a \). The converse is not true:

**EG:**

\[
 f(x, y) = \begin{cases} 
 x & \text{if } y = 0 \\
 y & \text{if } x = 0 \\
 0 & \text{otherwise}
\end{cases}
\]

\[
 \frac{\partial f}{\partial x}(0, 0) = 1
\]

\[
 \frac{\partial f}{\partial y}(0, 0) = 1
\]

\( f \) not cts at \((0, 0)\)

\( \therefore \) not differentiable at \((0, 0)\).

What is the \( ? \)

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↑ End Lec 7