Lecture 7

~~Recall~~

Recall Chain rule

$\boxed{\text{DO THIS LATER}}$

$\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n \qquad V \underset{\text{open}}{\subseteq} \mathbb{R}^m$

~~$f(\mathcal{U})$~~ $f : \mathcal{U} \to \mathbb{R}^M \qquad f(\mathcal{U}) \subseteq V$

$g : V \to \mathbb{R}^k$

$g \circ f : \mathcal{U} \to \mathbb{R}^k$

f diff'ble at a $\qquad g$ diff'ble at $f(a)$

~~$f \circ g$~~ $g \circ f$ diff'ble at a

$$(g \circ f)'(a) = g'(f(a)) \circ f'(a).$$

Two special cases of differentiability

① $f : (a-\varepsilon, a+\varepsilon) \longrightarrow \mathbb{R}$,

we usually define $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.

$$\Leftrightarrow \lim_{h \to 0} \frac{|f(a+h) - f(a) - f'(a)h|}{|h|} = 0.$$

The linear function $f'(a) : \mathbb{R} \to \mathbb{R}$ is $\times$ by the number $f'(a)$.

<u>Ex</u> Every linear function $L : \mathbb{R} \to \mathbb{R}$ is of the form

$$L(t) = \alpha t \quad \text{for } \forall t$$

for some $\alpha \in \mathbb{R}$.

② $f:(a-\varepsilon, a+\varepsilon) \longrightarrow \mathbb{R}^m$

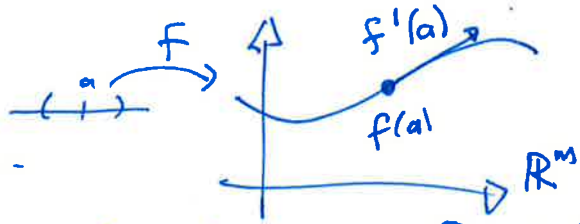
Can also define

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \in \mathbb{R}^m$$

again get $\iff \displaystyle\lim_{h\to 0} \frac{\|f(a+h)-f(a)-f'(a)h\|}{|h|}=0$

again the vector $f'(a)$ is the linear funct

$$\mathbb{R} \longrightarrow \mathbb{R}^m$$
$$h \longmapsto h\cdot f'(a).$$



Again Ex : Every linear function $L:\mathbb{R}\to\mathbb{R}^m$ has the form $L(t) = t\cdot v$ for some $v\in\mathbb{R}^m$.

Example

In particular if $a, v \in \mathbb{R}^m$ & $v\in\mathbb{R}^m$ &

$\phi(t) = a + tv$ $\qquad \phi:\mathbb{R}\to\mathbb{R}^m$

Then $\phi'(t) = v.$ $\quad \forall t$

Propⁿ 2.11 $f: U \longrightarrow \mathbb{R}^m$ & $v \in \mathbb{R}^n$

diff'ble at a

then $$f'(a)(v) = \frac{d}{dt}\left(f(a+tv)\right)\Big|_{t=0}$$

Proof Let $\phi(t) = a + tv$ $\qquad \phi(0) = a$

From Prop 2.5 $\quad \phi'(0) = v$

From Chain rule

$$(f \circ \phi)(t) = f(\phi(t)) = f(a+tv)$$

is diff'ble at 0 & f is diff'ble at a

& $$\frac{d}{dt}\left(f(a+tv)\right)\Big|_{t=0} = f'(a)\left(\phi'(0)\right)$$
$$= f'(a)(v).$$

Corollary 2.12 Let f be as above and let $f = (f^1, \dots, f^m)$. Let

$e^i = (0 \cdots 1 \cdots 0)$ then
$\qquad\qquad\uparrow$
$\qquad\quad$ ith place

$$f'(a)(e^i) = \left(\frac{\partial f^1}{\partial x^i}(a), \dots, \frac{\partial f^m}{\partial x^i}(a)\right)$$

(7.4)

$$\therefore \quad \cancel{\text{def}}\ f'(a)(v) = \begin{bmatrix} \frac{\partial f^1}{\partial x^1}(a) & \cdots & \frac{\partial f^1}{\partial x^n}(a) \\ & \vdots & \\ \frac{\partial f^m}{\partial x^1}(a) & \cdots & \frac{\partial f^m}{\partial x^n}(a) \end{bmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

$J(f)$

$$= J(f)(a)$$

Jacobian matrix

where $\frac{\partial f^j}{\partial x^i}(a) \underset{\text{def}^n}{=} \frac{d}{dt}\left(f^j(a+te^i)\right)\Big|_{t=0}$

Proof From Prop 2.3 we know that

$$\frac{d}{dt}\left(f(a+te^i)\right)_{t=0} = \left(\frac{df^1}{dt}(a+te^i)\Big|_{t=0}, \ \ldots \ \frac{df^m}{dt}(a+te^i)\Big|_{t=0}\right)$$

But by def

~~From~~ From Prop 2.11 $\quad f'(a)(e^i) = \frac{d}{dt} f(a+te^i)\Big|$

Then $\cancel{d}$ $f'(a)(v) = f'(a)\left(\sum_{i=1}^n v^i e^i\right)$

$$= \sum_{i=1}^n v^i \left(\frac{\partial f^1}{\partial x^i}(a), \frac{\partial f^2}{\partial x^i}(a) \ldots, \frac{\partial f^m}{\partial x^i}(a)\right)$$

$$= \big[J(f)\big]\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \quad \text{as req'd.}$$

Notice that we have shown that if f is diff'ble at a all the partial derivatives of f exist at a.

2.1 Functions of class C^k

<u>Def^n 2.13</u> Let $U \subseteq \mathbb{R}^n$ be open a $f : U \to \mathbb{R}$. We say that f is <u>of class C^k on U</u> if all partial derivatives of f on U up to order ~~up to exist~~ including k exist and are continuous.

We write C^0 for continuous functions & C^∞ for so-called smooth functions: funct that are of class C^k for all k.

We denote by $C^k(U)$ the set of all functions of class C^k on U.

<u>E.G.</u> (1) $f \in C^2(\mathbb{R}^2)$ if

$$f, \frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial^2 f}{(\partial x^1)^2}, \frac{\partial^2 f}{\partial x^1 \partial x^2}, \frac{\partial^2 f}{\partial x^2 \partial x^1}, \frac{\partial^2 f}{\partial x^{2}{}^2}$$

all exist and are cts.

(2) Any polynomial is in $C^\infty(\mathbb{R})$.
As are $\sin(x)$, $\cos(x)$, e^x, etc.

(3) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f(x_1, x_2) = \{ x_1^2 + \sin(x_2^2) \cancel{\ldots}$$

is in $C^\infty(\mathbb{R}^2)$

If $f : \mathcal{U} \to \in \mathbb{R}^m$ we say f is of class C^k or just C^k if $f = (f^1, \ldots, f^m)$ and each f^i is C^k. We write $f \in C^k(\mathcal{U}, \mathbb{R}^m)$.

<u>Lemma 2.14</u> Let $a < b < c$ &

(1) $f : (a, c) \to \mathbb{R}$ cts

(2) $\frac{\partial f}{\partial x} : f : (a, b) \cup (b, c) \to \mathbb{R}$ ~~and~~ diff'ble &

$$\lim_{x \to b^+} \frac{\partial f}{\partial x}(x) = \lim_{x \to b^-} \frac{\partial f}{\partial x}(x)$$

Then f is C^1 on (a, c)

Proof

Consider MVT on $[b, c]$

$$\frac{f(b+t) - f(b)}{t} = f'(\xi) \qquad b < \xi < t$$

as $t \to 0$ $\quad \xi \to \quad \lim_{t \to \cancel{b}0^+} \frac{f(b+t) - f(b)}{t} = f' \quad \lim_{\xi \to b^+} f'(\xi)$

Same from below. $\therefore$

$$\lim_{t \to 0} \frac{f(b+t) - f(b)}{t} = \lim_{\cancel{t} \to b^+} f'(\xi)$$

$$= \lim_{\xi \to b^-} f'(\xi)$$

$\therefore$ f diff'ble at b. f' cts on (a, c). //
& $\cancel{\frac{\partial f}{\partial x}}$ $f'(b) = \cancel{\lim}\ \lim_{\xi \to b} f'(\xi)$ $\therefore$

Example

Let $f_p(x) = \begin{cases} x^p & x \geq 0 \\ 0 & \leq 0 \end{cases}$ $\qquad p > 0 \qquad p \in \mathbb{R}, \ p \notin \mathbb{Z}$

$f^{(p)} = \begin{cases} p! \\ 0 \end{cases}$ $\qquad f_p^{(k)}(x) = \begin{cases} p(p-1)\cdots(p-k+1)\, x^{p-k} & x \geq 0 \\ 0 & \leq 0 \end{cases}$

$f_p \in C^0, C^1, \ldots, C^{p-1}$ but not C^p $\quad \therefore C^p \subsetneq C^{p+1}$

~~If $q < p$ then $f \in C^q$ by lemma~~

~~If $q \geq p$ then $f \notin C^q$~~ (~~check~~ p ?)

~~$f^{(p)}(x) = \begin{cases} p! & x \geq 0 \\ 0 & x \leq 0 \end{cases}$~~ not cts $\therefore$ ~~not diff~~'ble

We have seen that if f is diff'ble at a its partials are defined at a.

The converse is not true:

EG:

$$f(x,y) = \begin{cases} x & \text{if } y=0 \\ y & \text{if } x=0 \\ 1 & \text{o/wise} \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = 1$$

$$\frac{\partial f}{\partial y}(0,0) = 1$$

f not cts at $(0,0)$

$\therefore$ not diff'ble at $(0,0)$.

What is true?

End Lec 7