Lecture 6

Correction to Prop 2.3

$f : U \to \mathbb{R}^m \qquad f(x) = (f^1(x), \dots, f^m(x))$

f diff'ble at $a \iff f^i$ diff'ble at a $\forall i = 1, \dots, m$.

~~Re~~ & $f'(a)(v) = (f^{1\prime}(a)(v), \dots, f^{m\prime}(a)(v))$

~~Recall $f'(a) : \mathbb{R}^n \to \mathbb{R}^m$ linear so~~

$\cancel{f'(a)(v) = (L^1(v)}$ ($f'(a) : \mathbb{R}^n \to \mathbb{R}^m$ linear)
$\therefore f'(a)(v) \in \mathbb{R}^m$)

Ex $\quad L : \mathbb{R}^n \to \mathbb{R}^m \qquad L(v) = (L^1(v), \dots, L^m(v))$

$L^i : \mathbb{R}^n \to \mathbb{R}$

L linear $\iff L^1, \dots, L^m$ linear

In the proof we need $\quad w = (w^1, \dots, w^m) \in \mathbb{R}^m$

$$|w^i| \le \|w\| \le \sqrt{m}\, \max\{|w^i|\}$$

$$\|w\| = \|(w^1, \dots, w^m)\|$$

component of the vector

For example:

$$\Big|\underbrace{\big(f(a+h) - f(a) - f'(a)(h)\big)^i}\Big| \le \|f(a+h) - f(a) - f'(a)(h)\|$$

$$= f^i(a+h) - f^i(a) - (f'(a)(h))^i$$

$$\therefore \lim_{h \to 0} \frac{| f^i(a+h) - f^i(a) - (f'(a)(h))^i |}{\|h\|} = 0$$

f^i diff'ble at a &

$$f^{i\prime}(a)(h) = (f'(a)(h))^i$$

Similarly for the other directn

(2.11)

Prop 2.9 If $f, g : \mathcal{U} \to \mathbb{R}$ are diff'ble at $a \in \mathcal{U}$ the fg is diff'ble at a & $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$

proof

FIX UP COMPONENT THING

We have

$$\boxed{f(a+h)g(a+h)} - f(a)g(a)$$
$$- f(a)g'(a)(h) \quad - g(a)\, f'(a)(h)$$
$$f(a)g(a+h) \quad - \boxed{f(a)g(a+h)}$$
$$\boxed{-g(a+h)f'(a)h} + g(a+h)f'(a)h$$

$\therefore$

$$0 \le \| f(a+h)g(a+h) - f(a)g(a) - (f(a)g'(a)(h) + g(a)f'(a)(h)) \|$$
$*$

$$\le \|f(a)\| \, \| g(a+h) - g(a) - g'(a)(h) \|$$
$$+ \|g(a+h)\| \, \| f(a+h) - f(a) - f'(a)(h) \|$$
$$+ \|g(a+h) - g(a)\| \, \|f'(a)\| \, \|h\|$$

Divide by $\|h\|$

$$0 \le \frac{*}{\|h\|} \le \|f(a)\| \underbrace{\frac{\|g(a+h) - g(a) - g'(a)(h)\|}{\|h\|}}_{\to 0}$$

(2.12)

$$+ \|g(a+h)\| \frac{\|f(a+h) - f(a) - f'(a)(h)\|}{\|h\|} \to 0$$

$g(a)$

$$+ \|g(a+h) - g(a)\| \, \|f'(a)\|$$

$\downarrow$ 0

g is cts at a

g is cts at a

Prp 2.7

Prpⁿ 2.10 (Chain rule) $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$

Let $f : U \to \mathbb{R}^m$ $\quad V \underset{\text{open}}{\subseteq} \mathbb{R}^m$

$g : V \to \mathbb{R}^k$

$f(U) \subseteq V$

If f is diff'ble at a & g is diff'ble at $f(a)$ then $g \circ f : U \to \mathbb{R}^k$ is diff'ble at a &

$$(g \circ f)'(a) = g'(f(a)) \circ f'(a)$$

$$\left[\begin{array}{c} \mathbb{R}^m \xrightarrow{f'(a)} \mathbb{R}^n \xrightarrow{g'(f(a))} \mathbb{R}^k \\ \\ \begin{array}{ccccc} U & \xrightarrow{f} & V & \xrightarrow{g} & \mathbb{R}^k \\ \cap\! | & & \cap\! | & & \\ \mathbb{R}^n & & \mathbb{R}^m & & \end{array} \end{array} \right]$$

(2.9.3)

Case(i) Assume $g'(f(a)) = 0$ then consider:

what do we expect $(gf)'(a)$ to be?

$$\frac{\| gf(a+h) - g(f(a))\|}{\|h\|}$$

Let $\varepsilon > 0$. Choose $\delta > 0$ so that if $\|k\| < \delta$

$$\| g(f(a)+k) - g(f(a))\| \le \left(0 + \frac{\varepsilon}{\|f'(a)\|+1}\right)\|k\|$$

Choose $\delta' > 0$ so that

$\delta' < \dfrac{\delta}{\|f'(a)\|+1}$ & if $\|h\| < \delta'$ then

$$\| f(a+h) - f(a)\| \le (\|f'(a)\|+1)\|h\|$$

Let $k = f(a+h) - f(a)$.

If $\|h\| < \delta'$ then $\|k\| \le (\|f'(a)\|+1)\|h\| < \delta$

$$\therefore \| g(f(a+h)) - g(f(a))\| \le \frac{\varepsilon}{\|f'(a)\|+1}\|k\|$$

$$\le \varepsilon\|h\|$$

$$\lim_{h\to 0} \frac{\|g(f(a+h)) - g(f(a))\|}{\|h\|} = 0$$

$\therefore$ (gf) is diff'able at a & $(gf)'(a) = 0$
$= g'(f(a)) \circ f'(a)$

Case (2) $g(y) = L(y)$ L linear ?? Why

$$\frac{\| g f(a+h) - g f(a) - L f'(a) h \|}{\|h\|}$$

$$= \frac{\| L (f(a+h) - f(a) - f'(a)h) \|}{\|h\|}$$

$$\leq \|L\| \frac{\| f(a+h) - f(a) - f'(a) h \|}{\|h\|} \to 0$$

as $h \to 0$

$\therefore$ $gf'(a)\cancel{h} = L f'(a)(h)$

$= (g'(f(a)) \circ f'(a))(h)$.

General case

(derivative of linear $L = L$!)

Let $\tilde{g}(x) = g(x) - \underbrace{g'(f(a))}_{L}(x)$

then $\tilde{g}'(f(a)) = g'(f(a)) - g'(f(a)) = 0$

$\therefore$ ~~$\tilde{g}$ is diff'able at a~~ &

$\therefore$ ~~$\tilde{g}f(a)$~~ $\tilde{g} \circ f$ is diff'ble at a

& $(\tilde{g} \circ f)'(a) = 0$

But $\tilde{g} \circ f = g \circ f - g'(f(a)) \circ f$ &

(2.11)

$g'(f(a)) \circ f$ is diff'ble by case (2) ✓

$\therefore\ g \circ f = \tilde{g} \circ f + g'(f(a)) \circ f$ is diff'ble at a
Prop 2.8

& $(g \circ f)'(a) = 0 + g'(f(a)) \cdot f'(a)$. by case (2)
Prop 2.8

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End lecture 6