

Lecture 6

Correction to Prop 2.3

$$f: U \rightarrow \mathbb{R}^m \quad f(x) = (f^1(x), \dots, f^m(x))$$

f diff'ble at $a \iff f^i$ diff'ble at $a \forall i = 1, \dots, m$

Rec & $f'(a)(v) = (f^{1'}(a)(v), \dots, f^{m'}(a)(v))$

Recall ~~$f'(a): \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear~~

~~$f'(a)(v) \in \mathbb{R}^m$~~ | $f'(a): \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear
 $\therefore f'(a)(v) \in \mathbb{R}^m$

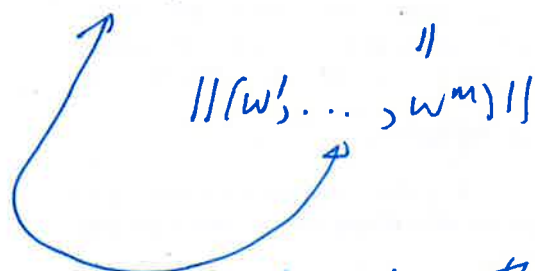
Ex $L: \mathbb{R}^n \rightarrow \mathbb{R}^m \quad L(v) = (L^1(v), \dots, L^m(v))$

$$L^i: \mathbb{R}^n \rightarrow \mathbb{R}$$

L linear $\iff L^1, \dots, L^m$ linear

In the proof we need $w = (w^1, \dots, w^m) \in \mathbb{R}^m$

$$|w^i| \leq \|w\| \leq \sqrt{m} \max \{ |w^i| \}$$



component of the vector

For example:

$$\underbrace{|(f^i(a+h) - f^i(a) - f^i{}'(a)h)|}_{= f^i(a+h) - f^i(a) - (f^i{}'(a)h)^i} \leq \|f(a+h) - f(a) - f'(a)h\|$$

$$\therefore \lim_{h \rightarrow 0} \frac{|f^i(a+h) - f^i(a) - (f^i(a)/h)^i|}{\|h\|} = 0$$

f^i diff'ble at a &

$$\underline{f^{i'}(a)(h) = (f^i(a)/h)^i}$$

Similarly for the other direction

(2.11)

Prp 2.9

If $f, g: U \rightarrow \mathbb{R}$ are diff'ble at $a \in U$
the fg is diff'ble at a & $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
proof

we have

FIX UP COMPONENT
THING

$$\frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$- f(a)g'(a)h$$

$$- g(a)f'(a)h$$

$$f(a)g(a+h)$$

$$- \left[\frac{f(a)g(a+h)}{h} \right]$$

$$- g(a+h)f'(a)h$$

$$+ g(a+h)f'(a)h$$

$$0 \leq \frac{\| f(a+h)g(a+h) - f(a)g(a) - (f(a)g'(a)h + g(a)f'(a)h) \|}{h}$$

$$\leq \|f(a)\| \|g(a+h) - g(a) - g'(a)h\|$$

$$+ \|g(a+h)\| \|f(a+h) - f(a) - f'(a)h\|$$

$$+ \|g(a+h) - g(a)\| \|f'(a)\| \|h\|$$

Divide by $\|h\|$

$$0 \leq \frac{*}{\|h\|} \leq \|f(a)\| \frac{\|g(a+h) - g(a) - g'(a)h\|}{\|h\|}$$

(2.12)

$$+ \|g(a+h)\| \frac{\|f(a+h) - f(a) - f'(a)h\|}{\|h\|}$$

$g(a)$ ↙
 g is cts at a
Prop 2.7

$$+ \frac{\|g(a+h) - g(a)\|}{\|h\|} \|f'(a)h\|$$

↓
 g is cts at a

Propⁿ 2.10 (Chain rule) $U \subseteq \mathbb{R}^n$
 U open

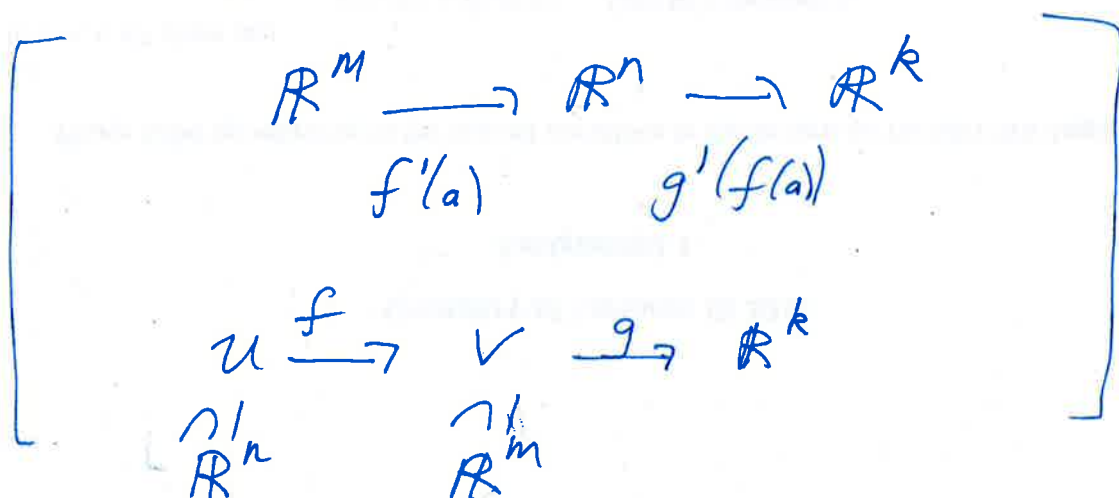
let $f: U \rightarrow \mathbb{R}^m$ $V \subseteq \mathbb{R}^m$
 V open

$g: V \rightarrow \mathbb{R}^k$

$f(U) \subseteq V$

⚡ If f is diff'ble at a & g is diff'ble at $f(a)$ then $g \circ f: U \rightarrow \mathbb{R}^k$ is diff'ble at a &

$$(g \circ f)'(a) = g'(f(a)) \circ f'(a)$$



(2.93)

Case 1)

Assume $g'(f(a)) = 0$

what do we expect $(gf)'(a)$ to be?
then consider:

$$\frac{\|g(f(a+h)) - g(f(a))\|}{\|h\|}$$

Let $\varepsilon > 0$. Choose $\delta > 0$ so that if $\|k\| < \delta$

$$\|g(f(a)+k) - g(f(a))\| \leq \left(0 + \frac{\varepsilon}{\|f'(a)\|+1}\right) \|k\|$$

Choose $\delta' > 0$ so that ~~if~~

$$\delta' < \frac{\delta}{\|f'(a)\|+1} \quad \& \quad \text{if } \|h\| < \delta' \text{ then}$$

$$\|f(a+h) - f(a)\| \leq (\|f'(a)\|+1) \|h\|$$

Let $k = f(a+h) - f(a)$

$$\text{If } \|h\| < \delta' \text{ then } \|k\| \leq (\|f'(a)\|+1) \|h\| < \delta$$

$$\begin{aligned} \therefore \|g(f(a+h)) - g(f(a))\| &\leq \frac{\varepsilon}{\|f'(a)\|+1} \|k\| \\ &\leq \varepsilon \|h\| \end{aligned}$$

$$\lim_{h \rightarrow 0} \frac{\|g(f(a+h)) - g(f(a))\|}{\|h\|} = 0$$

$\therefore (gf)$ is differentiable at a & $(gf)'(a) = 0$
 $= g'(f(a)) \cdot f'(a)$

(2.10)

Case (2)

$$g(y) = L(y) \quad L \text{ linear}$$

Why

$$\|g(f(a+h)) - g(f(a)) - L(f'(a)h)\|$$

$$\|h\|$$

$$= \frac{\|L(f(a+h) - f(a) - f'(a)h)\|}{\|h\|}$$

$$\leq \frac{\|L\| \|f(a+h) - f(a) - f'(a)h\|}{\|h\|} \rightarrow 0 \quad \text{as } h \rightarrow 0$$

$$\begin{aligned} \therefore g(f'(a)h) &= L(f'(a)h) \\ &= (g'(f(a)) \circ f'(a))(h) \end{aligned}$$

General case

L!

(derivative of linear $L=L'$)

$$\text{let } \tilde{g}(x) = g(x) - g'(f(a))(x)$$

$$\text{then } \tilde{g}'(f(a)) = g'(f(a)) - g'(f(a)) = 0$$

~~\tilde{g} is differentiable at a~~ &

~~$\tilde{g}(a)$~~ $\tilde{g} \circ f$ is differentiable at a

$$\& \quad (\tilde{g} \circ f)'(a) = 0$$

$$\text{But } \tilde{g} \circ f = g \circ f - g'(f(a)) \circ f \quad \&$$

(2.11)

$g'(f(a)) \circ f$ is diff'ble by case (2) ✓

$\therefore g \circ f = \tilde{g} \circ f + g'(f(a)) \circ f$ is diff'ble at a
Prop 2.8

& $(g \circ f)'(a) = 0 + g'(f(a)) \circ f'(a)$ by case (2)
Prop 2.8

↓ End lecture 6
