Lecture 5 (2.4)

* Room change ?
+ Assignment Monday

lim

$\mathbb{R}^n$
U open
$f : U \longrightarrow \mathbb{R}^m$
$L : \mathbb{R}^n \to \mathbb{R}^m$ linear

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - L(h) \|}{\| h \|} = 0$$

$$L = f'(a)$$

vector in $\mathbb{R}^m$ vector in $\mathbb{R}^n$

$$\therefore \; f(x) \sim f(a) + f'(a)(x-a)$$

<u>linear approximation</u>

linear map $\mathbb{R}^n \to \mathbb{R}^m$ vector in $\mathbb{R}^n$

If f is diff'ble at all $a \in U$
we say <u>f is differentiable on U</u>

~~Lectures 5~~ 2.5 ~~Def of~~ linear app

RECALL Def^n of derivative / at a

~~So if f is differentiable, we denote the L in the definition by $f'(a)$~~

~~If f is diff'ble at $a \in \mathcal{U}$~~ $\forall a \in \mathcal{U}$ say f diff'ble on $\mathcal{U}$.

Proposition 2.3

Let $\mathcal{U} \subseteq \mathbb{R}^n$ open $f : \mathcal{U} \to \mathbb{R}^m$. Write $f(x) = (f^1(x), \ldots, f^m(x))$ then f is differentiable at a if and only if f^i is differentiable at a for all i and $f'(a) = (f^{1\prime}(a), f^{2\prime}(a), \ldots, f^{m\prime}(a))$.

Proof ($\Leftarrow$)

We have

$$0 \leq \frac{\| f^i(a+h) - f^i(a) - f^{i\prime}(a)(h) \|}{\|h\|}$$

$$\leq \frac{\| f(a+h) - f(a) - f'(a)(h) \|}{\|h\|}$$

$\therefore$ f diff'ble at $a \Rightarrow f^{i\prime}$ diff'able at a $\forall$ ~~a.~~ i

f^i diff'able ~~at~~ at a $\forall i$:

But if, $\varepsilon > 0$ $\exists$ $\delta_i > 0$ s.t. $\|h\| < \delta_i$ $h \in \mathcal{U}$

$$\Rightarrow \frac{\| f^i(x+h) - f^i(x) - f^{i\prime}(x)(h) \|}{\|h\|} < \frac{\varepsilon}{\sqrt{n}}$$

So if $h < \delta = \min\{\delta_1, \ldots, \delta_m\}$ then

$$\max \frac{\| f^i(x+h) - f^i(x) - f^{i\prime}(x)(h) \|}{\|h\|} < \varepsilon/\sqrt{n}$$

$$\therefore \frac{\| f(x+h) - f(x) - f'(x)(h) \|}{\|h\|} < \varepsilon$$

$$\therefore \lim_{h \to 0} \frac{\| f(x+h) - f(x) - f'(x)h \|}{\|h\|} = 0$$

$\therefore f$ diff'ble at a. //

~~[illegible]~~ Example

$f : \mathbb{R}^3 \to \mathbb{R}^2$

$f(x, y, z) = \begin{pmatrix} x^2 + y \\ yz \end{pmatrix}$

$f^1(x, y, z) = x^2 + y$
$f^2(x, y, z) = yz$ } differentiable everywhere

$\therefore f$ differentiable everywhere

$a = (1, 0, 1)$

$f(1, 0, 1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$h = (\alpha, \beta, \gamma)$

$f(1+\alpha, 0+\beta, 1+\gamma) = \begin{pmatrix} (1+\alpha)^2 + \beta \\ (\beta)(1+\gamma) \end{pmatrix} = \begin{pmatrix} 1 + \alpha^2 + 2\alpha + \beta \\ \beta + \beta\gamma \end{pmatrix}$

$$f((1,0,1)+h) - f(1,0,1) = \begin{pmatrix} 2\alpha + \beta \\ \beta \end{pmatrix} + \begin{pmatrix} \alpha^2 \\ \beta\gamma \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} \alpha^2 \\ \beta\gamma \end{pmatrix}$$

$$\frac{\| \qquad \|^2}{\|h\|^2} = \left(\frac{\sqrt{\alpha^4+\beta^2\gamma^2}}{\sqrt{\alpha^2+\beta^2+\gamma^2}} \right)^2 = \frac{\alpha^4+\beta^2\gamma^2}{\alpha^2+\beta^2+\gamma^2}$$

$$\leq \frac{(\alpha^2+\beta^2+\gamma^2)^2}{(\alpha^2+\beta^2+\gamma^2)} = \|h\|^2 \longrightarrow 0 \quad \text{as } \|h\| \to 0 \,!$$

$$\delta f(a)\left(\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\right) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

$f'(a)$: $\mathbb{R}^3 \to \mathbb{R}^2$

cf to $f'(1,0,1)$

<u>Lemma 2.4</u> If $U \subseteq \mathbb{R}^n$ U open $f : U \to \mathbb{R}^m$ then f is differentiable at a iff $\exists$ a linear function L and $\varepsilon > 0$ and an $R : B(0,\varepsilon) \to \mathbb{R}^m$ such that

$$f(a+h) = f(a) + L(h) + R(h)$$

$$\& \quad \lim_{h\to 0} \frac{\|R(h)\|}{\|h\|} = 0$$

<u>Proof</u> Just let $R(h) = f(a+h) - f(a) - L(h)$ & result follows from def^n.

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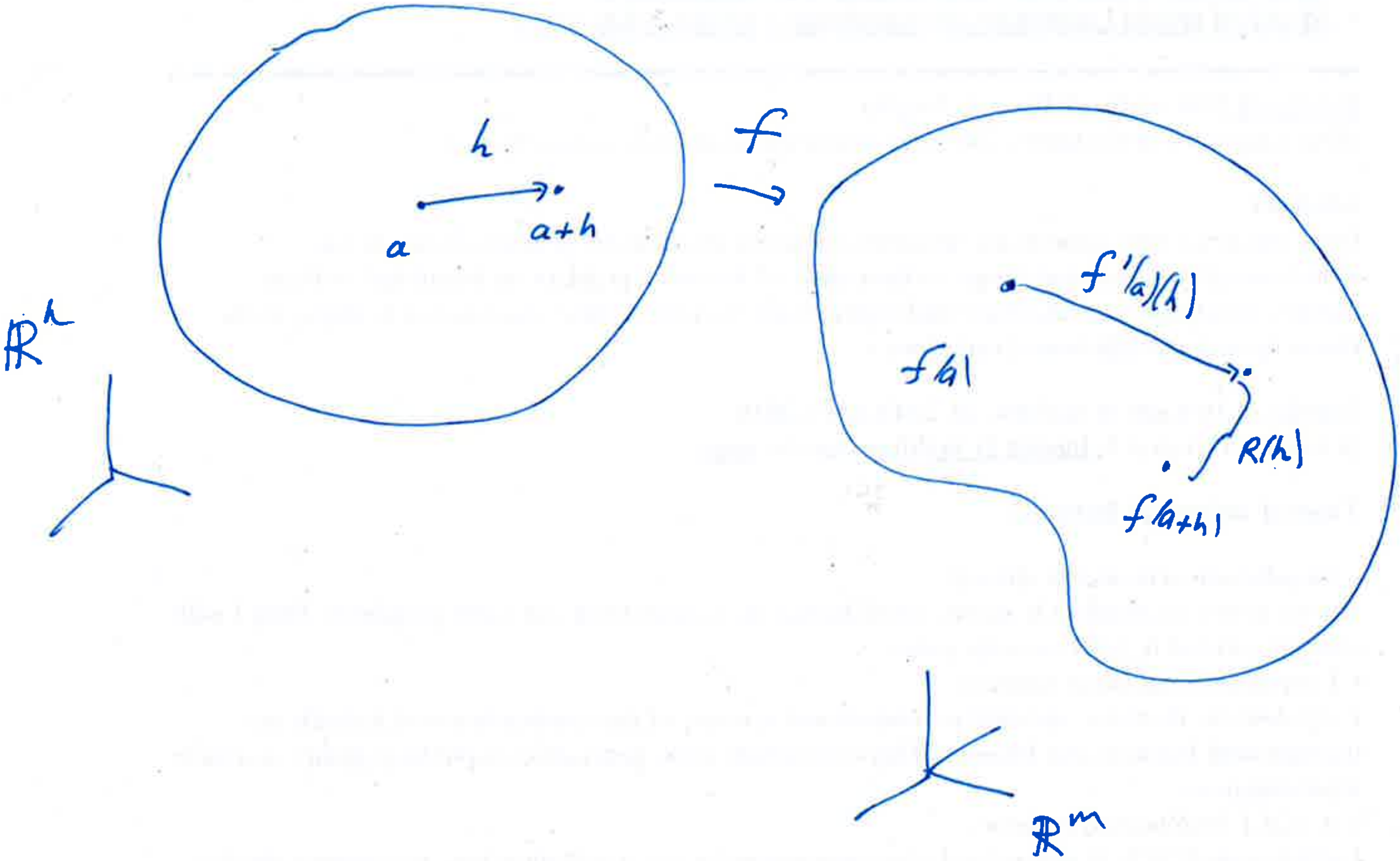
Eg in ex $R\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\alpha^2\\ \beta\gamma\end{pmatrix}$

<u>NB</u> <u>R depends on a</u>



Can also write: $f(x) = f(a) + L(x-a) + R(x-a)$

~~Lenjulat~~ $\& v \in \mathbb{R}^m$

Prop^n 2.5 If $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $f(x) = L(x) + v$ is differentiable at every $a \in \mathbb{R}^n$ and $f'(a) = L$.

Proof (Ex).

Lemma 2.6 If f is differentiable at a ($f : \mathcal{U} \to \mathbb{R}^m$, $\mathcal{U} \subset \mathbb{R}^n$) then $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t if $\|h\| < \delta$ then $\|f(a+h) - f(a)\| \leq (\|f'(a)\| + \varepsilon)\|h\|$.

Proof Choose $\delta > 0$ so that

$$\frac{\|f(a+h) - f(a) - f'(a)h\|}{\|h\|} < \varepsilon \quad \text{if } 0 \leq \|h\| < \delta$$

$\therefore$

$$\begin{aligned} &\|f(a+h) - f(a)\| \\ &\leq \|f(a+h) - f(a) - f'(a)h\| + \|f'(a)\|\|h\| \\ &\leq \varepsilon\|h\| + \|f'(a)\|\|h\| = (\|f'(a)\| + \varepsilon)\|h\| \end{aligned}$$

Prop^n 2.7 If f is diff'ble at a then f is cts at a.

Proof ~~then~~ Chose ε, δ as above then

$\forall \|h\| < \delta$

$$0 \leq \|f(a+h) - f(a)\| \leq (\|f'(a)\| + \varepsilon)\|h\|$$

$\therefore$ $\|f(a+h) - f(a)\| \to 0$ as $\|h\| \to 0$.

$\therefore$ $\lim_{h \to 0} f(a+h) = f(a)$ $\therefore$ f cts at a //

Prop^n 2.8

If $f, g : U \to \mathbb{R}^m$ are diff'ble at $a \in U$ ~~then~~ & $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g$ is diff'ble at $a \in U$ &

$$(\alpha f + \beta g)'(a) = \alpha f'(a) + \beta g'(a)$$

Proof

$$0 \le \lim_{h \to 0} \frac{\| (\alpha f + \beta g)(a+h) - (\alpha f + \beta g)(a) - (\alpha f'(a) + \beta g'(a))(h) \|}{\|h\|}$$

$$\le \lim_{h \to 0} |\alpha| \frac{\| f(a+h) - f(a) - f'(a)(h) \|}{\|h\|}$$

$$+ \lim_{h \to 0} |\beta| \frac{\| g(a+h) - g(a) - g'(a)(h) \|}{\|h\|}$$

$\to 0$ $\therefore$ limit exists. //

End lects