Lecture 4

(1.14)

Propn 1.14 (Properties of continuous functions)

① If $\mathcal{U} \subseteq \mathbb{R}^n$ open & $f, g : \mathcal{U} \to \mathbb{R}^m$, ~~a+b~~ cts & $\alpha, \beta \in \mathbb{R}$ then

$$\alpha f + \beta g : \mathcal{U} \to \mathbb{R}^m$$

is continuous

② If $f : \mathcal{U} \to \mathbb{R}^m$ $f(x) = (f^1(x), \dots, f^m(x))$

f is cts $\Leftrightarrow$ $f^1, \dots, f^m$ are all cts

③ Let $f : \mathcal{U} \to \mathbb{R}^m$ $g : V \to \mathbb{R}^k$ $V \subseteq \mathbb{R}^m$

$f(\mathcal{U}) \subseteq V$. Then if f and g are cts so also is

$$g \circ f : \mathcal{U} \longrightarrow \mathbb{R}^k$$

defined by $g \circ f(x) = g(f(x))$

Proof ① Let $a \in \mathcal{U}$ as f, g are cts $\lim_{x \to a} f(x) = f(a)$ & $\lim_{x \to a} g(x) = g(a)$.

From 1.10

$$\lim_{x \to a} \alpha f(x) + \beta g(x) = \alpha \lim_{x \to a} f(x) + \beta \lim_{x \to a} g(x)$$

$$= \alpha f(a) + \beta g(a) \quad (1.15)$$

So $\alpha f + \beta g$ is cts

(2) ~~Follows from 1.10~~ $\boxed{\text{Do}}$

(3) ~~Let~~ We use 1.13. Let $x_n \to a$ in $\mathcal{U}$.

Then $f(x_n) \to f(a)$ in V so $g(f(x_n)) \to gf(a)$ in $\mathbb{R}^k$

$\therefore$ $g \circ f(x_n) \longrightarrow g \circ f(a)$

$\therefore$ $g \circ f$ is cts at a.

$\therefore$ $g \circ f$ is cts in $\mathcal{U}$.

We need a special case of a thm called the Contraction Mapping Thm. Some of you will have seen this in T&A.

If $r>0$ let $\overline{B}(0,r) = \{y \in \mathbb{R}^n \mid \|y\| \leq r\}$

(closed ball of radius r)

A function $f : \overline{B}(0,r) \to \overline{B}(0,r)$ is called a contraction if $\exists\ 0 \leq K < 1$ s.t. $\forall x, y \in \overline{B}(0,r)$

$$\|f(x) - f(y)\| \leq K \|x - y\|$$

then we have

Prop^n 1.15 If $f : \overline{B}(0,r) \to \overline{B}(0,r)$ is a contraction then ~~[illegible]~~ there is a unique $x \in \overline{B}(0,r)$ s.t. $f(x) = x$. (a fixed point)

Proof Note that if $f(x) = x$ & $f(y) = y$ then ~~$\|f(x) - f(y)\| \leq K \|x$~~

$$\|x - y\| = \|f(x) - f(y)\| \leq K\|x - y\|$$

& $K < 1$ $\therefore$ $\|x - y\| = 0$. $\therefore$ $x = y$. So unique.

Choose $x_0 \in \overline{B}(0,r)$ & let $x_1 = f(x_0)$, $x_2 = f(x_1)$ etc.

Notice $\|x_{r+1} - x_r\| = \|f(x_r) - f(x_{r-1})\|$

$$= \le k\|x_r - x_{r-1}\| \qquad (1.17)$$

$$= k\|f(x_{r-1}) - f(x_{r-2})\|$$

$$\le k^2\|x_{r-1} - x_{r-2}\|$$

$$\vdots$$

$$\le k^r\|x_1 - x_0\|$$

$$\therefore \boxed{\|x_{r+1} - x_r\| \le k^r\|x_1 - x_0\|}$$

Let $n \geqslant m$ say $n = m+k$

$$\|x_n - x_m\| = \|x_{m+k} - x_m\|$$

$$\geqslant \|x_{m+k} - x_{m+k-1}\| + \dots + \|x_{m+1} - x_m\|$$

$$\geqslant (k^{m+k-1} + \dots + k^m)\|x_1 - x_0\|$$

$$= k^m(k^{k-1} + \dots + 1)\|x_1 - x_0\|$$

(As $0 \le k < 1$ then $1 + k + k^2 + \dots = \dfrac{1}{1-k}$)

$$\therefore \geqslant \frac{k^m}{1-k}\|x_1 - x_0\|$$

As $n, m \to \infty$ $\dfrac{k^m}{1-k}\|x_1 - x_0\| \to 0$ as $k < 1$ $\therefore (x_n)$ is Cauchy $\Rightarrow x_n \to x$ for some x.

$\dfrac{k^m}{1-k} \to \dfrac{0}{1-k}$

Where is x?

Not in general

Ass 1 $\lim\|x_n\| = \|x\|$

$\le \|y - z\| + \|z\|$

$\therefore$ ~~$\|y\| - \|z\|$~~ $\le \|y - z\|$ $\boxed{\text{Ass 1}}$

If $\|x\| > r$ then $\|x - x_m\| \to 0$

So we can find m s.t. $\|x_m\| > r$

a contradiction

$$\boxed{\therefore x \in \overline{B}(0,r)}$$

~~(1.1?)~~ (1.18)

Also $0 \le \| f(x) - f(x_m) \| \le K \| x - x_m \| \to 0$

$\therefore$ $f(x_m) \to f(x)$. (Squeeze Lemma)

$\parallel$

$x_{m+1} \to x$ $\therefore$ $f(x) = x$.

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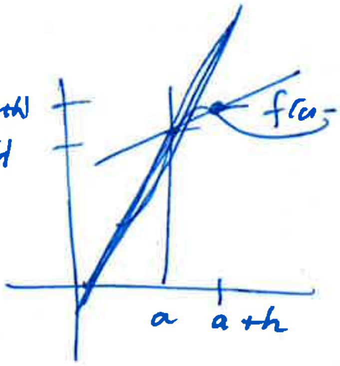
§2 Differentiation in $\mathbb{R}^n$

Recall that in $\mathbb{R}$ we define



$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This doesn't make sense if $f : \underset{\substack{\text{open} \\ \mathbb{R}^n}}{U} \to \mathbb{R}^m$

as h is a vector.

Can also think of <u>linear approximation</u>

$$f(a+h) = f(a) + f'(a)h + R(h)$$

$$\lim_{h \to 0} \frac{R(h)}{h} = 0$$

If $f : U \to \mathbb{R}^m$ $\quad \underset{\text{open}}{U \subseteq \mathbb{R}^n}$

$$f(a+h) = f(a) + f'(a)(h) + R(h)$$

make sense and we regard

$$f'(a) : \mathbb{R}^n \to \mathbb{R}^m$$

as linear (<u>matrix</u>)

In such a case we expect

$$\lim_{h \to 0} \frac{\|R(h)\|}{\|h\|} \to 0.$$

We come back to the idea in a moment but for now use it for

<u>Def^n 2.1</u> Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open & $f : \mathcal{U} \to \mathbb{R}^m$. We say that f is differentiable at $a \in \mathcal{U}$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0$$

~~Of~~

<u>Note</u> ~~that we don't have a formula~~

$$\cancel{f'(a)(h)} = \dots$$

1. Of course we expect L to be ~~...~~ $f'(a)$ but we don't know it is unique!

2. Note we don't have a formula

$$L(h) = \lim \dots$$

as in case of $f : \mathbb{R} \to \mathbb{R}$

Lemma 2.2 If L exists in defn 2.1 it is unique.

Proof Assume

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0$$

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - J(h)\|}{\|h\|} = 0$$

$$0 \leq \frac{\|L(h) - J(h)\|}{\|h\|} = \frac{\|(f(a+h) - f(a) - J(h)) - (f(a+h) - f(a) - L(h))\|}{\|h\|}$$

$$\leq \frac{\|f(a+h) - f(a) - J(h)\|}{\|h\|} + \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|}$$

$\to 0$ as $\|h\| \to 0$

Let $h = tu$ $\|u\| = 1$ then

$$0 = \lim_{t\to 0} \frac{\|L(tu) - J(tu)\|}{t} = \|L(u) - J(u)\|$$

$\therefore$ $L(u) = J(u)$ $\forall u$ $\|u\| = 1$

$\therefore$ $L(v) = J(v)$ $\forall v \in \mathbb{R}^n$

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End Lecture 4