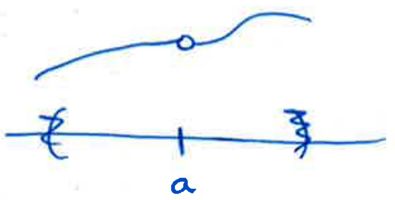
Lecture 3   29/7/2011   (1.8)

Review one-dimensional ca

$\boxed{\text{cf Maths 1}}$



$\lim_{x \to a} f(x) = L$

$\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0$ s.t.
$|x-a| \Rightarrow |f(x) - \cancel{f(a)} L| < \varepsilon$

Def^n 1.7 Let $a \in \mathcal{U} \underset{\text{open}}{\subset} \mathbb{R}^n$ and $f : \mathcal{U} - \{a\} \to \mathbb{R}^m$.
We say f has limit $L \in \mathbb{R}^m$ if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $x \in \mathcal{U}$ & $\|x - a\| < \delta$ then $\|f(x) - L\| < \varepsilon$
(OR: if $x \in \mathcal{U} \cap B(a, \delta)$ then $f(x) \in B(L, \varepsilon)$)
We write $\lim_{x \to a} f(x) = L$.

~~cf Maths 1~~

It is often simplest to work with

Prop^n 1.8 If $f : \mathcal{U} - \{a\} \to \mathbb{R}^m$ then

$\lim_{x \to a} f(x) = L$ if and only if $\forall x_n \to a$

we have $f(x_n) \to L$.

Using Prop^n 1.1 we deduce:
(Prop^n 1 – Componentwise)
Prop^n 1.6 2 If $f, g : \mathcal{U} - \{a\} \to \mathbb{R}^m$, $\lim_{x \to a} f(x) = L$

& $\lim_{x \to a} g(x) = J$ & $\alpha, \beta \in \mathbb{R}$ then

$$\lim_{x \to a} \alpha f(x) + \beta g(x) = \alpha L + \beta J.$$

Proof

($\Rightarrow$) Assume $\lim_{x \to a} f(x) = L$ & $x_n \to a$

as $n \to \infty$. Then let $\varepsilon > 0$ $\exists$

$\exists \delta > 0$ s.t. if $\|x - a\| < \delta$, & $x \in \mathcal{U}$

~~$\exists$~~ then $\|f(x) - L\| < \varepsilon$.

Def^n of limit

Choose N s.t. if $n \geq N$ then

$\|x_n - a\| < \delta$

then $\|f(x_n) - L\| < \varepsilon$

$\therefore$ $f(x_n) \to L$ as $n \to \infty$.

($\Leftarrow$) ~~Assume $x_n \to a$ & $f(x_n) \to L$~~

(Proof by contradiction) Assume

$\lim_{x \to a} f(x) \neq L$

So $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x_\delta \in \mathcal{U}$ &

$\|x_\delta - a\| < \delta$ & $\|f(x_\delta) - L\| > \varepsilon$

Let $\delta = \frac{1}{n}$ & let $x_n = x_\delta$

then $0 \leq \|x_n - a\| < \frac{1}{n}$ $\forall n$ $\lim_{n \to \infty} \frac{1}{n} = 0$ Squeeze Lemma

$\therefore$ $x_n \to a$ but $\|f(x_n) - L\| > \varepsilon$

$\therefore$ $f(x_n) \not\to L$ a contradiction. //

~~Prop^n~~ Lemma 1.9 (Squeeze Lemma)

Assume $f, g, h : \mathcal{U} - \{a\} \to \mathbb{R}^m$ ($\mathcal{U}$ open $\subseteq \mathbb{R}^n$)

& $f(x) \leq g(x) \leq h(x) \quad \forall x \in \mathcal{U} - \{a\}$

If $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$

then $\lim_{x \to a} g(x)$

Proof. ~~Let $x_n \to$~~ Use Squeeze Lemma for sequences. Let $x_n \to a$ the

$f(x_n) \leq g(x_n) \leq h(x_n)$

& $f(x_n) \to L$, $h(x_n) \to L$ (~~Squeeze Lemma~~ ~~Prop 1.8~~) (Prop 1.8)

$\therefore$ $g(x_n) \to L$ (Squeeze Lemma)

$\therefore$ $\lim_{x \to a} g = L$ (Prop 1.8). //

(1.80)

Prop^n 1.10 (Properties of limits)

(1) Let $f : U - \{a\} \to \mathbb{R}^m$ & write $f(x) = (f^1(x), \ldots, f^m(x))$. ~~then~~

Then $\lim_{x \to a} f(x) = L = (L^1, \ldots, L^m)$

$\Leftrightarrow$ $\lim_{x \to a} f^1(x) = L^1, \ldots, \lim_{x \to a} f^m(x) = L^m$.

(2) Let $f, g : U - \{a\} \to \mathbb{R}^m$ & $\alpha, \beta \in \mathbb{R}$

then $\lim_{x \to a} \alpha f(x) + \beta g(x) = \alpha \lim_{x \to a} f(x) + \beta \lim_{x \to a} g(x)$

Proof

(1) Prop 1.8 + Prop 1.4 HA

(2) " "

//

Defⁿ 1.11 Let $\mathcal{U} \underset{\text{open}}{\subset} \mathbb{R}^n$ & $f : \mathcal{U} \to \mathbb{R}^m$. We say that f is cts at $a \in \mathcal{U}$ if $\lim_{x \to a} f(x) = f(a)$ & cts on $\mathcal{U}$ if cts at all $a \in \mathcal{U}$.

~~We have:~~

~~**Propⁿ 1.8** A function f is cts at~~ a ~~if and only if $f(x)$ $\lim_{x_n \to a} f(x_n) = f(a)$ for all x: $\forall x_n \to a$.~~

~~(Follows from defⁿ & Propⁿ 1.5)~~

Propⁿ 1.12 A function is cts at $a \iff \lim_{x \to a} \|f(x) - f(a)\| = 0$ (REAL NUMBER CASE)

Prop 1.13 f cts at $a \iff \forall x_n \to a$ $f(x_n) \to f(a)$ ~~(Properties of cts function)~~ (Def & 1.8)

Example. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be linear.

$$L(x) = \begin{bmatrix} - & L_1 & - \\ - & L_2 & - \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

(1.12)

$$= (\langle L_1, x\rangle, \ldots \langle L_m, x\rangle)$$

$$\|L(x)\|^2 = \sum_{i=1}^{m} |\langle L_i, x\rangle|^2$$

$$\leq \left(\sum_{i=1}^{m} \|L_i\|^2\right) \|x\|^2$$

$\therefore \quad \|L(x)\| \leq C \|x\|$ $\quad C$ a constant not depending on x.

Then $\|L(x) - L(y)\| ~~\leq C \|x -~~$

$$= \|L(x-y)\| \leq C \|x - y\|$$

If $\varepsilon > 0$ let $\delta = \varepsilon / C$ $\therefore$

$$\|x - a\| < \delta \Rightarrow \|L(x) - L(a)\|$$

$$\leq C\delta = \varepsilon.$$

$\therefore \lim_{x \to a} L(x) = L(a)$. $\quad \therefore L$ ct at all a.

$\therefore$ L ct everywhen

Notice that here we have used

$$C = \sqrt{\sum_{i=1}^{m} \|L_i\|^2}$$

this is a "norm" on matrices. ~~A~~ The more common one is to define

$$\|L\| = \max \left\{ \|L(x)\| \;\middle|\; \begin{matrix} x \in \mathbb{R}^n \\ \|x\| = 1 \end{matrix} \right\}$$

THIS IS DIFFERENT TO C !

However it also satisfies

$$\|L(x)\| \leq \|L\| \, \|x\|$$

Proof : If $x = 0$ we are done.

If $x \neq 0$ $\quad \left\| \frac{x}{\|x\|} \right\| = 1 \quad \therefore$

$$\|L\| \geq \left\| L\left(\frac{x}{\|x\|}\right) \right\|$$

$$= \|L(x)\| \frac{1}{\|x\|}$$

$$\therefore \quad \|L(x)\| \leq \|L\| \, \|x\|$$

Usually if we write $\|L\|$ we mean this one.

End lec 3