Lecture 2 1.3 27/7/2011

Recall (x_n) a sequence in $\mathbb{R}^n$ has a limit $x \in \mathbb{R}^n$ if $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n \geq N$ $\|x_n - x\| < \varepsilon$

Note: ① $x_n \in B(x, \varepsilon)$: ~~event~~ $\forall n \geq N$ DRAW PICTURE

② Can ~~[illegible]~~ replace ε by $\frac{1}{10^k}$ for any k then if x_n, x are real numbers $\|x_n - x\| < \frac{1}{10^k}$ means x_n & x agree up to the kth decimal place.

Do $x_i \to 0$?

x_1
x_2
x_3
$\vdots$
x

$\underbrace{\qquad}_{k \text{ places}}$ never change

Lemma 1.1

If (x_n) has a ~~un~~ limit it is unique

Proof

Assume $\lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} x_n = y$

then

$$0 \leq \|x-y\| = \|x - x_n + x_n - y\|$$
$$= \|(x - x_n) + (x_n - y)\|$$
$$\leq \|x - x_n\| + \|y - y_n\|$$

But we can make RHS as small as we like. but $\|x-y\|$ is constant so $\|x-y\| = 0$ $\therefore$ $x = y$. //

Lemma 1.2

A sequence (x_n) has limit x if and only if the sequence of real numbers $\|x_n - x\|$ has limit 0.

ie $\lim_{n \to \infty} \|x_n - x\| = 0$

Proof

Just the definition. //

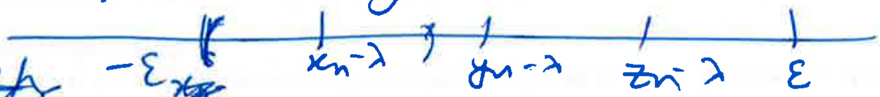
The argument in Lemma 1.1 is going to be common. It is an application of the Squeeze Lemma

Lemma 1.3 (Squeeze Lemma)

~~Let~~ (x_n) (y_n) (z_n) be sequences in $\mathbb{R}$ with $x_n \le y_n \le z_n$. If $x_n \to x$ & $z_n \to x$ then $y_n \to x$

Proof Draw picture

We have $x_n - x \le y_n - x \le z_n - x$



Let $\varepsilon > 0$ & choose N_1 s.t. $|x_n - x| < \varepsilon$ $\forall n \geq N_1$. Choose N_2 s.t. $|z_n - x| < \varepsilon$

(1.5) $\boxed{}$

Prop^n 1.4 (Property of limit of seq)

② If $x_n \to x$ & $y_n \to y$

$$\alpha x_n + \beta y_n \to \alpha x + \beta y$$

Proof ①

~~① $x_m - x = (x^1, \ldots, x^n)$ $\Leftrightarrow x_m^i \to x^i$~~

For $|x_m^i - x^i| \le \|x_m - x\|$ ✓

⇐ Let $\varepsilon > 0$ choose N_i s.t. for $\forall m \geq N_i$, $|x_m^i - x^i| < \varepsilon/\sqrt{n}$. Then let $N = \max\{N_1, \ldots\}$

$\therefore \forall i$, $\forall m \geq N$ $(\geq N_i)$

$|x_m^i - x^i| < \varepsilon/\sqrt{n}$

$\therefore \|x_m - x\| \le \sqrt{n} \max(\;) < \varepsilon$

② $0 \le \|\alpha x_n + \beta y_n - (\alpha x + \beta y)\|$

$$\le |\alpha|\,\|x_n - x\| + |\beta|\,\|y_n - y\| \longrightarrow 0$$

$$\therefore \|(\alpha x_n + \beta y_n) - (\alpha x + \beta y)\| \to 0$$

(Squeeze Lemma)

Lemma 1.5 If $x_n \to x$ then $\lim_{m,n\to\infty} \|x_n - x_m\| = 0$

Proof

$$0 \le \|x_m - x_n\| = \|x_m - x_n + (x - x)\| = \|(x_m - x) - (x_n - x)\|$$

$$\le \|x_m - x\| + \|x_n - x\|$$

$$\longrightarrow 0 \quad \text{as } m, n \to \infty$$

Def We call a sequence (x_n) Cauchy if $\lim_{m,n\to\infty} \|x_n - x_m\| = 0$

(c.f. testing convergence in a computer algorithm)

(1.5)

$\forall n > N_2$. Let $N = \max\{N_1, N_2\}$. So if $n \geq N$ then $n \geq N_1$ & $n \geq N_2$ and

$$-\varepsilon < x_n - x < \varepsilon \quad (\Leftrightarrow |x_n - x| < \varepsilon)$$

&

$$-\varepsilon < z_n - x < \varepsilon.$$

$$\therefore \quad -\varepsilon < x_n - x \leq y_n - x \leq z_n - x \leq \varepsilon$$

$$\therefore \quad -\varepsilon \leq y_n - x < \varepsilon$$

$$\therefore \Rightarrow \quad |y_n - x| < \varepsilon.$$

$$\therefore \quad \lim_{n\to\infty} y_n = x.$$

We often use this in the form

$$0 \leq x_n \leq y_n \quad \& \quad y_n \to 0$$

$$\therefore \quad x_n \to 0.$$

Propn 1.4

(i) If $x_m = (x_m^1, \dots, x_m^n) \in \mathbb{R}^n$

then $x_m \to x = (x^1, \dots, x^n)$ if and only if

$$\Leftrightarrow \begin{cases} x_m^1 \to x^1 \quad \cancel{\text{as } m\to\infty} \\ x_m^2 \to x^2 \\ \vdots \\ x^n \to x^n. \end{cases}$$

Thm 1.6 (Completeness of $\mathbb{R}^n$)

Every ~~[illegible]~~ Cauchy sequence in $\mathbb{R}^n$ converges.

Proof Follows from $n=1$ case. ~~For~~ ~~[illegible]~~ $\mathbb{R}$ it is the lub axiom.

EXPLAIN

$\boxed{\text{Recall } \mathbb{R} \text{ case, } f'(x) \text{ etc}}$

Defn 1.7 Let $a \in \underset{\text{open}}{U \subset} \mathbb{R}^n$ and $f : U - \{a\} \to \mathbb{R}^m$.

We say f has limit $L \in \mathbb{R}^m$ if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $x \in U$ & $\|x - a\| < \delta$ then $\|f(x) - L\| < \varepsilon$

We write $\lim_{x \to a} f(x) = L$.

cf Maths I

It is often simplest to work with

Propn 1.8 If $f : U - \{a\} \to \mathbb{R}^m$ then $\lim_{x\to a} f(x) = L$ if and only if $\forall x_n \to a$ we have $f(x_n) \to L$.

Using Propn 1.1 we deduce:

(Propn ① Componentwise)

Propn 1.6 ② If $f, g : U - \{a\} \to \mathbb{R}^m$, $\lim_{x\to a} f(x) = L$ & $\lim_{x \to a} g(x) = J$ & $\alpha, \beta \in \mathbb{R}$ then

$$\lim_{x\to a} \alpha f(x) + \beta g(x) = \alpha L + \beta J.$$