Recall

\[ \eta = \eta_1 \, d\hat{x}^1 + \eta_2 \, d\hat{x}^2 \]

\[ = \, \tilde{\eta}_1 \, d\hat{x}^1 + \tilde{\eta}_2 \, d\hat{x}^2 \]

Then:

\[ d\eta = \left( \frac{\partial \eta_2 \circ \chi}{\partial \hat{x}^1} - \frac{\partial \eta_1 \circ \chi}{\partial \hat{x}^2} \right) \, d\hat{x}^1 \wedge d\hat{x}^2 \]

Also:

\[ = \left( \frac{\partial \tilde{\eta}_2 \circ \chi}{\partial \hat{x}^1} - \frac{\partial \tilde{\eta}_2 \circ \chi}{\partial \hat{x}^2} \right) \, d\hat{x}^1 \wedge d\hat{x}^2 \]

We showed:

\[ d\hat{x}^j = \sum_{i=1}^{2} \frac{\partial x^\circ \circ \psi_j}{\partial \hat{x}^i} \, d\hat{x}^i \]

\[ \frac{\partial \eta_2 \circ \chi}{\partial \hat{x}^1} = \sum_{l,m} \frac{\partial \tilde{\eta}_0 \circ \chi}{\partial \hat{x}^2} \frac{\partial (x^\circ \chi)^l}{\partial \hat{x}^1} \frac{\partial (x^\circ \chi)^m}{\partial \hat{x}^2} \\
+ \sum_m \tilde{\eta}_0 \circ \chi \frac{\partial^2 (x^\circ \chi)^m}{\partial \hat{x}^1 \partial \hat{x}^2} \]

& \mathbb{2} \Rightarrow 1

Sub into RHS of \( \bigcirc \). Second term vanishes because symmetric in 1 \( \leftrightarrow \) 2.
Let $u = hv + tp$.

If $h = \frac{1}{v}$ then $u = \frac{1}{v} - \frac{5}{6}$. If $7 = \frac{1}{v}$ then $u = \frac{1}{v} - \frac{3}{2}$.

As a parameterization $x = \frac{e}{\sqrt{2}}$,

$\frac{dv}{\sqrt{2}} = dv$. 

where $v$ is the denominator of $x$.

With $v = 1$,

$\text{denv} = 0$. 

$\frac{dv}{\sqrt{2}}$.
Prop 6.23 (Weak Green's Thm)

If $\eta$ is a 1-form on $\Sigma \subseteq \mathbb{R}^3$ oriented surface, then

$$\int_{\Sigma} d\eta = 0$$

Proof

Choose a partition of unity $\rho_{\alpha}$ for a cover by parametrizations $\mathcal{U}_{\alpha}$.

Let

$$\eta = \sum_{\alpha} \rho_{\alpha} \eta_{\alpha}$$

Then

$$d\eta = \sum_{\alpha} d(\rho_{\alpha} \eta_{\alpha})$$

Hence

$$\int_{\Sigma} d\eta = \sum_{\alpha} \int_{\mathcal{U}_{\alpha}} d(\rho_{\alpha} \eta_{\alpha})$$

Since

$$\text{supp}(\rho_{\alpha} \eta_{\alpha}) \subseteq \mathcal{U}_{\alpha} \cap (\mathcal{U}_{\alpha} \setminus \mathcal{U}_{\beta})$$

Then

$$\text{supp} (d(\rho_{\alpha} \eta_{\alpha})) \subseteq \mathcal{U}_{\alpha} \setminus \mathcal{U}_{\beta}$$

Therefore

$$\int_{\Sigma} d\eta = \sum_{\alpha} \int_{\mathcal{U}_{\alpha} \setminus \mathcal{U}_{\beta}} d(\rho_{\alpha} \eta_{\alpha})$$
So it is enough to show \( \int_{\gamma_0(U \cup U')} d(\rho \chi \eta) = 0 \),

\( \forall \mu \in \mathcal{M}_c(U) \cap \mathcal{M}_c(U') \).

On \( U \subset \mathbb{R}^2 \) open, bounded \&

\( \eta \) a 1-form with supp \( \eta \subseteq U \) then

\[ \int_{\eta} d\eta = 0 \]

But \( \eta = \gamma_1 \, dx^1 + \gamma_2 \, dx^2 \)

Greens Thm \( \Rightarrow \)

\[ \int_{\eta} d\eta = \int \left( \frac{\partial \gamma_2}{\partial x^1} - \frac{\partial \gamma_1}{\partial x^2} \right) dx^1 \, dx^2 \]

\[ = \int_0 \gamma_1 \, dx^1 + \gamma_2 \, dx^2 \]

\[ = 0 \]
Proposition 7.1

\[
\frac{d}{dt} \left( R_{\Sigma} v^I_{\Sigma} t \right) = d \eta
\]

Proof

We can work locally to prove equality.

Assume \( \Psi(x^1, x^2) \) local coordinates parameters as \( \Sigma \) but suppress their dependence. Then we have

\[
\eta = \left< \frac{\partial \eta}{\partial t} \times \frac{\partial \eta}{\partial x^1}, \eta \right> d \Psi^1 + \left< \frac{\partial \eta}{\partial t} \times \frac{\partial \eta}{\partial x^2}, \eta \right> d \Psi^2
\]

\[
d\eta = \left( \left< \frac{\partial^2 \eta}{\partial x^1 \partial t} \times \frac{\partial \eta}{\partial x^1}, \eta \right> + \left< \frac{\partial \eta}{\partial t} \times \frac{\partial^2 \eta}{\partial x^1 \partial x^2}, \eta \right> \right) + \left< \frac{\partial \eta}{\partial t} \times \frac{\partial \eta}{\partial x^2}, \eta \right>
\]

\[
= \left( \left< \frac{\partial^2 \eta}{\partial x^1 \partial t} \times \frac{\partial \eta}{\partial x^1}, \eta \right> - \left< \frac{\partial^2 \eta}{\partial x^2 \partial t} \times \frac{\partial \eta}{\partial x^2}, \eta \right> \right) d \Psi^1 \wedge d \Psi^2
\]
Note from earlier work that
\[ R u d (x, y) = \langle \pi(x) \times \pi(y), n \rangle \]

\[ = \langle n'(x) \times n'(y), n \rangle \]

So we can calculate it entirely using the Gauss map.

Imagine now we deformed \( \Sigma' \) in \( \mathbb{R}^6 \)

we want to see how \( \int R u d \) changes \( \Sigma \)

If we can instead hold \( \Sigma' \) fixed and

just assume \( n = n_t \) is varying.

\[ \eta : [0,1] \times \Sigma' \rightarrow S^2 \]

we define a 1-form \( \eta \) on \( \Sigma' \) as

\[ \eta(x) = \langle \frac{dn}{dt} \times dn(x), n \rangle \]
\[
\frac{\partial}{\partial t} \langle \frac{\partial \eta}{\partial x^1} \times \frac{\partial \eta}{\partial x^2}, \eta \rangle = \frac{\partial}{\partial t} (Rudl)
\]

Theorem 7.2

\[\int_{R} v \, d\mathcal{E}\] is constant if we deform \(\Sigma\).

Proof

\[
\frac{\partial}{\partial t} \int_{R} v \, d\mathcal{E} = \int \frac{\partial v}{\partial t} \, d\mathcal{E} = 0.
\]

Definition 7.3

If \(\tilde{\Sigma}\) is obtained from \(\Sigma\) by removing two disks and adding a cylinder, we call it attaching a handle."