Lecture 27

Recall

$$\eta = \eta_1 \, d\hat{\psi}^1 + \eta_2 \, d\hat{\psi}^2$$
$$= \tilde{\eta}_1 \, d\hat{\chi}^1 + \tilde{\eta}_2 \, d\hat{\chi}^2$$

Def.
$$d\eta = \left(\frac{\partial \eta_2 \circ \psi}{\partial x^1} - \frac{\partial \eta_1 \circ \psi}{\partial x^2} \right) d\hat{\psi}^1 \wedge d\hat{\psi}^2 \qquad (*)$$

also
$$\overset{?}{=} \left(\frac{\partial \tilde{\eta}_2 \circ \chi}{\partial x^1} - \frac{\partial \tilde{\eta}_2 \circ \chi}{\partial x^2} \right) d\hat{\chi}^1 \wedge d\hat{\chi}^2 \quad ??$$

We showed

(1)
$$d\hat{\chi}^j = \sum_{i=1}^{2} \frac{\partial (\chi^{-1} \circ \psi)^j}{\partial x^i} \, d\psi^i$$

(2)
$$\frac{\partial \eta_2 \circ \psi}{\partial x^1} = \sum_{l,m} \frac{\partial \tilde{\eta}_m \circ \chi}{\partial x^l} \underset{(2)}{\frac{\partial (\chi^{-1} \circ \psi)^l}{\partial x^1}} \underset{(1)}{\frac{\partial (\chi^{-1} \circ \psi)^m}{\partial x^2}}$$
$$+ \sum_m \tilde{\eta}_m \circ \psi \, \frac{\partial^2 (\chi^{-1} \circ \psi)^m}{\partial x^1 \partial x^2}$$

& $2 \leftrightarrow 1$

Sub into RHS of $(*)$. Second term vanishes because symmetric in $1 \leftrightarrow 2$.

Leaving

$$\sum_{l,\dots} \frac{\partial \tilde{\eta}_m \circ X}{\partial x^l} \frac{\partial (X^{-1}\circ\psi)^l}{\partial x^1} \frac{\partial (X^{-1}\circ\psi)^m}{\partial x^2} \, d\hat{\psi}^1 \wedge d\hat{\psi}^2$$

!

$$+ \sum \frac{\partial \tilde{\eta}_m \circ X}{\partial x^l} \frac{\partial (X^{-1}\circ\psi)^l}{\partial x^2} \frac{\partial (X^{-1}\circ\psi)^m}{\partial x^1} \, d\hat{\psi}^2 \wedge d\hat{\psi}^1$$

Swapped

$$\cancel{= \sum_{l,m} \left(\frac{\partial \tilde{\eta}_m \circ X}{\partial x^l} - \frac{\partial \tilde{\eta}_l \circ X}{\partial x^m} \right) d\hat{\psi}^l \wedge d\hat{\psi}^m}$$

$$= \sum_{l,m} \frac{\partial \tilde{\eta}_m \circ X}{\partial x^l} \, d\hat{\psi}^l \wedge d\hat{\psi}^m$$

$$= \left(\frac{\partial \tilde{\eta}_2 \circ X}{\partial x^1} - \frac{\partial \tilde{\eta}_1 \circ X}{\partial x^2} \right) d\hat{\psi}^1 \wedge d\hat{\psi}^2$$

$(d\mu \wedge d\mu = 0$ - ex$)$.

Prop^n 6.22

① If $\psi : U \to \Sigma'$ is a parametrisation & η a 1-form then

$$d\eta(s) = \sum \left(\frac{\partial \eta_2 \circ \psi}{\partial x^1}(\psi^{-1}(s)) - \frac{\partial \eta_1 \circ \psi}{\partial x^2}(\psi^{-1}(s)) \right) d\hat{\psi}^1(s) \wedge d\hat{\psi}^2(s)$$

is well-defined &

② If $f : \Sigma \to \mathbb{R}$ then

$$d(f \wedge \eta) = df \wedge \eta + f \wedge d\eta$$

27/3

Proof ① ✓ ② Ex //

Prop^n 6.23 (Weak Green's Th^m)

If η is a 1-form on $\Sigma \subseteq \mathbb{R}^3$ oriented cl/surface

$$\int_\Sigma d\eta = 0$$

Proof Choose a partition of unity ρ_α for a cover by parametrisation ψ_α.

$$\eta = \sum_\alpha \rho_\alpha \eta$$

$$\therefore \; d\eta = \sum_\alpha d(\rho_\alpha \eta)$$

$$\int_\Sigma d\eta = \sum_\alpha \int_\Sigma d(\rho_\alpha \eta)$$

$\mathrm{supp}(\rho_\alpha \eta) \subseteq \psi_\alpha(U_\alpha)$

$$\therefore \; \mathrm{supp}(d(\rho_\alpha \eta)) \subseteq \psi_\alpha(U_\alpha)$$

$$\therefore \int_\Sigma d\eta = \sum_\alpha \int_{\psi_\alpha(U_\alpha)} d(\rho_\alpha \eta)$$

So it is enough to show $\displaystyle\int_{\psi_\alpha(U_\alpha)} d(\rho_\alpha \eta) = 0.$

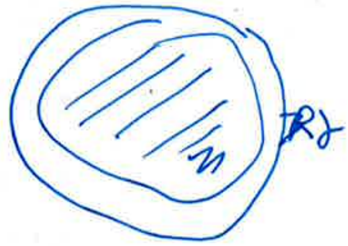
or if $U \subseteq \mathbb{R}^2$ open bounded & η a 1-form with $\operatorname{supp} \eta \subseteq U$ then

$$\int_U d\eta = 0$$

But $\eta = \eta_1 \, dx^1 + \eta_2 \, dx^2$

Green's Thm $\Rightarrow$

$$\int_U d\eta = \int_{R_\gamma} \left(\frac{\partial \eta_2}{\partial x^1} - \frac{\partial \eta_1}{\partial x^2} \right) dx^1 dx^2$$



$$= \int_\gamma \eta_1 \, dx^1 + \eta_2 \, dx^2$$

$$= 0. \quad //$$

Prop^n 7.1

$$\frac{d}{dt}\left(R_t \, v \, \delta_t \right) = d\eta$$

Proof

We can work locally ~~on~~ to prove equality.

Assume $\psi(x^1, x^2)$ local ~~co-ordinates for~~ parameters as Σ' but suppress their dependence. Then

$$\eta = \left\langle \frac{\partial n}{\partial t} \times \frac{\partial n}{\partial x^1}, n \right\rangle d\hat{\psi}^1 + \left\langle \frac{\partial n}{\partial t} \times \frac{\partial n}{\partial x^2}, n \right\rangle d\hat{\psi}^2$$

$$d\eta = \Bigg(\left\langle \frac{\partial^2 n}{\partial x^1 \partial t} \times \frac{\partial n}{\partial x^2}, n \right\rangle + \left\langle \frac{\partial n}{\partial t} \times \frac{\partial^2 n}{\partial x^1 \partial x^2}, n \right\rangle + \underbrace{\left\langle \frac{\partial n}{\partial t} \times \frac{\partial n}{\partial x^2}, \frac{\partial n}{\partial x^1} \right\rangle}_{} \Bigg) d\hat{\psi}^1 \wedge d\hat{\psi}^2$$

Symmetric in 1–2

$= 0$ as all $\perp n$

$$- \left(\quad 1 \longleftrightarrow 2 \quad \right)$$

$$= \left(\left\langle \frac{\partial^2 n}{\partial x^1 \partial t} \times \frac{\partial n}{\partial x^2}, n \right\rangle - \left\langle \frac{\partial^2 n}{\partial x^2 \partial t} \times \frac{\partial n}{\partial x^1}, n \right\rangle \right) d\hat{\psi}^1 \wedge d\hat{\psi}^2.$$

§7 GAUSS-BONNET THEOREM

Note from earlier work that

$$R\nu d\,(X, Y) = \langle \pi(X) \times \pi(Y), n \rangle$$

$$= \langle n'(X) \times n'(Y), n \rangle$$

& we can calculate it entirely using the Gauss map.

Imagine now we deformed Σ' in $\mathbb{R}^3$

We want to see how $\int_\Sigma R\nu d$ changes

& we can instead hold Σ' fixed and just assume $n = n_t$ is varying.

~~[illegible]~~ $\therefore$ $n : [0,1] \times \Sigma' \longrightarrow S^2$

We define a 1-form η on Σ' as

$$\eta(X) = \langle \frac{\partial n}{\partial t} \times dn(X), n \rangle$$

$= \cancel{\frac{d}{dt}}$

$= \frac{\partial}{\partial t}\left(\langle \frac{\partial n}{\partial x^1} \times \frac{\partial n}{\partial x^2}, n \rangle \right) = \frac{\partial}{\partial t}(Rv\partial)$

<u>Corollary</u> 7.2

$\int R\, v d_{\Sigma^t}$ is constant wt one

deform Σ^t.

<u>Proof</u> $\frac{d}{dt} \int R v d_{\Sigma} = \int d\eta = 0.$

<u>Def 7.3</u> If $\tilde{\Sigma}^t$ is obtained from Σ^t

by removing two disks and adding

a cylinder we call it <u>attaching a</u>

<u>handle</u>