Lecture 26

Recall $V^* =$ all linear maps $V \to \mathbb{R}$

= one-forms.

$\dim V = 2$ $\quad w^1, w^2 \in V^*$ define

$w^1 \wedge w^2 \in \det(V^*)$ by

$$(w^1 \wedge w^2)(v^1, v^2) = w^1(v^1)w^2(v^2) - w^1(v^2)w^2(v^1)$$

If $S \subseteq \mathbb{R}^N$ is a submanifold &

$f : S \to \mathbb{R}$ is smooth then $f'(s) : T_sS \to \mathbb{R}$

is a 1-form. From now on call it df

or $df(s)$. <u>the exterior derivative ~~of~~ f.</u>

Let $\psi : \mathcal{U} \to S$ be a parametrisation.

Then $\psi^{-1} : \psi(\mathcal{U}) \to \mathbb{R}^n$ ~~&~~ ~~$\hat\psi = (\hat\psi^1, \ldots$~~

Denote it by $\hat\psi = \psi^{-1}$ and let $\hat\psi = (\hat\psi^1, \ldots, \hat\psi^n)$

where $\hat{\psi}^i : \psi(U) \to \mathbb{R}$.

Hence we have $d\hat{\psi}^i(s) : T_s S \to \mathbb{R}$ linear

<u>Prop^n 6.19</u> Assume $\psi(x) = s$ then the one-forms $d\hat{\psi}^1(s), \dots, d\hat{\psi}^n(s)$ are dual to the basis $\frac{\partial \psi}{\partial x^1}(x), \dots, \frac{\partial \psi}{\partial x^n}(x)$.

<u>Proof</u>
$$d\hat{\psi}^i(s)\left(\frac{\partial \psi}{\partial x^j}(x)\right) = \frac{d}{dx^j}\left(\hat{\psi}^i \circ \psi\right)\Big/_x$$

$$\frac{d}{dt}\left(\hat{\psi}^i \psi(x + te^j)\right) = \frac{d}{dt}(x + te^j)^i = \delta^{ij}.$$

It follows from linear algebra (Ex) that they form a basis. //

<u>Def^n 6.20</u> A one-form η on S a submanifold in $\mathbb{R}^N$ is a choice for each s of a one-form $\eta(s) : T_s S \to \mathbb{R}$. For each s $\eta(s) = \sum_{i=1}^{n} \eta_i(s)\, d\hat{\psi}^i(s)$

and we call η <u>smooth</u> if ~~...~~ we can cover S by image of params such that the η_i are smooth

<u>Note</u>: It can be checked that ~~the~~ smoothness of η is independent of the choice of parameters

we do this as follows

Assume $\psi : V \to S$ $\quad \chi : V \to S$ $\quad \chi(u) = \psi(u)$

Then $\cancel{\frac{\partial \psi}{\partial x^i}(x) = \psi'(x)(e_i) = \psi^2 \circ \chi^{-1}}$

$\cancel{\psi = \chi \circ \chi^{-1} \circ \psi}$

$\cancel{\therefore \frac{\partial \psi}{\partial x^i}(x) = \psi'(x)(e_i) = \chi'(\chi^{-1}\circ\psi(x))\,(\chi^{-1}\circ\psi)'(x)(e_i)}$

$\cancel{= \sum_j \frac{\partial \chi}{\partial x^j}(\ ) \frac{\partial(\chi^{-1}\circ\psi)^j}{\partial x^i}(x)}$

seen before that $\quad \boxed{\text{Lecture 24}}$

$$\boxed{\frac{\partial \psi}{\partial x^i}(x) = \sum_{j=1}^{n} \frac{\partial (\chi^{-1}\circ\psi)^j}{\partial x^i}(x) \; \frac{\partial \chi}{\partial x^j}\left(\chi^{-1}\circ\psi(x)\right)}$$

$\cancel{d\psi^i(x) \text{ is}}$

$\cancel{d\hat{\psi}^i(\psi(x)) \text{ is equal to } \frac{\partial \psi}{\partial x^i}(x)}$

$\therefore \quad \cancel{d\hat{\chi}^j(\chi(x)) = \sum \frac{\partial(\chi^{-1}\circ\psi)^j}{\partial x^i}(x)}$

$$\eta = \sum \eta_i(s)\, d\hat{\psi}^i(s) \qquad \psi(x) = s$$

$$= \sum \tilde{\eta}_j(s)\, d\hat{\chi}^j(s) \qquad \chi(\chi^{-1}\circ\psi(x)) = s$$

$\boxed{\text{dual basis}}$

$$\eta_i(s) = \eta\left(\frac{\partial \psi}{\partial x^i}(x)\right)$$

$$= \sum \frac{\partial(\chi^{-1}\circ\psi)^j}{\partial x^i}(x)\; \underbrace{\eta\left(\frac{\partial \chi}{\partial x^j}\left(\chi^{-1}\circ\psi(x)\right)\right)}_{= \tilde{\eta}_j(s)}.$$

$\therefore \tilde{\eta}_j(s)$ C^∞

$\Longrightarrow \eta_i(s)$ C^∞

Example $\quad f: S \to \mathbb{R}$

$df(s): T_s S \to \mathbb{R}$

Notice that if $\eta = \sum \eta_i \, d\hat{\varphi}^i$

then $\eta_i = \eta\left(\frac{\partial \varphi}{\partial x^i}\right)$ from dual basis

relation.

$$\therefore \quad (df)(s) = \sum df\left(\frac{\partial \varphi}{\partial x^i}\right) d\hat{\varphi}^i(s)$$

$$= \sum \frac{\partial f \circ \varphi}{\partial x^i}(x) \, d\hat{\varphi}^i(s)$$

if $\varphi(x) = s$.

Lemma 6.~~4~~21

$$d\hat{\varphi}^1 \wedge d\hat{\varphi}^2 = \left[\frac{\partial \varphi}{\partial x^1}, \frac{\partial \varphi}{\partial x^2}\right]$$

Proof Let v_1, v_2 be a basis of 2-dim^l V

v_1^*, v_2^* the dual basis

$$(v_1^* \wedge v_2^*)\left(\sum w_i v_i, \sum u_i v_i\right) = w_1 u_2 - w_2 u_1$$

$$[v_1 v_2]\left(\sum w_i v_i, \sum u_i v_i\right) = w_1 u_2 - w_2 u_1 \quad //$$

Differentiating one-form

Let $\eta = \eta_1 \, d\hat{\varphi}^1 + \eta_2 \, d\hat{\varphi}^2$

be a 1-form. ~~We define~~ a a surface.

We define

$$d\eta(s) = \left(\frac{\partial \eta_2 \circ \varphi}{\partial x^1}(\varphi^{-1}(s)) - \frac{\partial \eta_1 \circ \varphi}{\partial x^2}(\varphi^{-1}(s)) \right) d\hat{\varphi}^1(s) \wedge d\hat{\varphi}^2(s)$$

a 2-form

I claim this doesn't chan if we change parameters

$\varphi : U \to S$

$\chi : V \to S$

Assume

$$\boxed{\eta = \tilde{\eta}_1 \, d\hat{\chi} + \tilde{\eta}_2 \, d\hat{\chi}}$$

Now $\frac{\partial \varphi}{\partial x^i}(\varphi^{-1}(s)) = \sum \frac{\partial (\chi^{-1} \circ \varphi)^j}{\partial x^i}(\varphi^{-1}(s)) \frac{\partial \chi}{\partial x^j}(\chi^{-1}(s))$

$\therefore$ ~~$\eta_i(s) = \eta\left(\frac{\partial \varphi}{\partial x^i}(\varphi^{-1}(s))\right) = \sum \eta\left(\frac{\partial \chi}{\partial x^j}(\chi^{-1}(s)) \frac{\partial (\chi^{-1} \circ \varphi)^j}{\partial x^i}(\varphi^{-1}(s))\right)$~~

$$\therefore \boxed{\eta_i(s) = \sum \tilde{\eta}_j(s) \frac{\partial (\chi^{-1} \circ \varphi)^j}{\partial x^i}(\varphi^{-1}(s))}$$

already done

and $$d\hat{\chi}^j(s) = \sum \frac{\partial(\chi^{-1}\circ\psi)^j}{\partial x^i}(\varphi^{-1}(s))\; d\hat{\psi}^i(s)$$

(Follows by duality from relation for tangent vectors.)

Then

$$\frac{\partial \eta_i \circ \psi}{\partial x^j}(\varphi^{-1}(s)) = \sum \frac{\partial}{\partial x^j}\left(\tilde{\eta}_m \circ \psi \; \frac{\partial(\chi^{-1}\circ\psi)^m}{\partial x^i}(\psi^{-1}(s))\right)$$

$$= \sum \frac{\partial \tilde{\eta}_m \circ \chi}{\partial x^l}(\chi^{-1}(s)) \frac{\partial(\chi^{-1}\circ\psi)^l}{\partial x^j}(x) \quad \frac{\partial(\chi^{-1}\circ\psi)^m}{\partial x^i}(x)$$

$$+ \sum (\tilde{\eta}_m \circ \psi)(\psi^{-1}(m)) \frac{\partial^2(\chi^{-1}\circ\psi)^m}{\partial x^j \partial x^i}(x)$$