Recall \( \mathbf{V}^* = \text{all linear maps } \mathbf{V} \to \mathbb{R} \) are forms.

\( \dim \mathbf{V} = 2 \), \( \mathbf{w}^1, \mathbf{w}^2 \in \mathbf{V}^* \) define

\( \mathbf{w}^1 \mathbf{w}^2 \in \text{det} (\mathbf{V}^*) \) by

\[
(\mathbf{w}^1 \mathbf{w}^2)(\mathbf{v}^1, \mathbf{v}^2) = \mathbf{w}^1(\mathbf{v}^1) \mathbf{w}^2(\mathbf{v}^2) - \mathbf{w}^1(\mathbf{v}^2) \mathbf{w}^2(\mathbf{v}^1)
\]

If \( \mathbb{S} \subseteq \mathbb{R}^n \) is a submanifold and \( f : \mathbb{S} \to \mathbb{R} \) is smooth, then \( f'(s) : T_s \mathbb{S} \to \mathbb{R} \)

is a linear form. From now on call it \( df \) or \( df(s) \). The exterior derivative of \( f \).

Let \( \Psi : U \to \mathbb{S} \) be a parametrization.

Then \( \Psi^{-1} : \Psi(U) \to \mathbb{R}^n \) & \( \Psi^{-1}(\mathbf{y}) = (y_1, \ldots, y_n) \).

Denote it by \( \mathbf{y} = \Psi^{-1} \) and let \( \mathbf{y}^k = (y^k, \ldots, \mathbf{y}^n) \)
where \( \hat{\gamma} : \gamma(u) \to \mathbb{R} \).

Here we have \( \frac{d\hat{\gamma}}{du}(s) : T_S S \to \mathbb{R} \) linear.

Def \( 6.18 \) Assume \( \gamma(x) = s \) then the one-family \( \hat{\gamma}(s), \ldots, \hat{\gamma}^n(s) \) are dual to the basis \( \frac{\partial \gamma}{\partial x_1}(x), \ldots, \frac{\partial \gamma}{\partial x_n}(x) \).

Proof

\[
\frac{d\hat{\gamma}}{du}(s) \left( \frac{\partial \gamma}{\partial x_i}(x) \right) = \frac{d}{du} \left( \hat{\gamma} \circ \gamma \right) / x
\]

\[
d\left( \hat{\gamma} \left( \gamma(x + te^i) \right) \right) = \frac{d}{dt} \left( \gamma(x + te^i) \right) = \delta_{ij}.
\]

It follows from linear algebra (Ex) that they form a basis.

Def \( 6.20 \) A one-family \( \gamma \) is a 
submanifol
\( u \in \mathbb{R}^n \) is a choice for each \( s \) of a one-to-
\( \gamma : T_S S \to \mathbb{R} \). For each \( s \) \( \gamma(s) = \sum_{i=1}^{n} \hat{\gamma}^i(s) \frac{\partial \gamma}{\partial x_i}(s) \)

and we call \( \gamma \) smooth if \( \hat{\gamma}^i \) are smooth.

We can choose \( s \) by image of parameter such that \( \hat{\gamma}^i \) are smooth.

It can be checked that the smoothness of \( \gamma \) is independent of the choice of parameters.
we do this as follows

Assume \( \psi: V \to S \) \( \chi: V \to S \) \( \chi(x) = \psi(y) \)

Then

\[
\begin{align*}
\frac{\partial y}{\partial x_1}(x) &= y'(1)(x_1) = x'(1)(y'(1)(x_1)) = x'(1)(\psi'(x))
\end{align*}
\]

seen before that

\[
\begin{align*}
\frac{\partial y}{\partial x_i}(x) &= \sum_{j=1}^{n} \frac{\partial (\chi^{-1} \psi(x))}{\partial x_i}(x) \frac{\partial \chi}{\partial x_j}(x) \psi'(x)
\end{align*}
\]

\[
\begin{align*}
\eta_i(s) &= \sum_{j=1}^{n} \eta_j'(s) \psi'(x)
\end{align*}
\]

\( \eta_i(s) \in C^\infty \)

\[
\begin{align*}
\eta_i'(s) &= \eta_i'(s) \left( \frac{\partial \psi}{\partial x_i}(x) \right)
\end{align*}
\]

\[
\begin{align*}
\bar{\eta}_j(s) &= \sum_{i=1}^{n} \bar{\eta}_i(s) \left( \frac{\partial \chi}{\partial x_i}(x) \right) \eta_i'(s)
\end{align*}
\]
Example \( f : S \to \mathbb{R} \)

\[ df(x) : T_x S \to \mathbb{R} \]

Notice that \( \eta = \sum_{i} \eta_{i} \, d\psi_{i} \)

then \( \eta_{i} = \eta \left( \frac{\partial \psi}{\partial x_{i}} \right) \) have dual basis

\[ (df)_{x} = \sum_{i} df \left( \frac{\partial \psi}{\partial x_{i}} \right) \, d\psi_{i}(x) \]

\[ = \sum_{i} \frac{df \circ \psi}{dx_{i}}(x) \, d\psi_{i}(x) \]

\[ \eta_{i}(\phi_{i}) = s . \]

Lemma 6.21

\[ \int d\psi_{i} \, d\psi_{j} = \left[ \frac{\partial \psi}{\partial x_{i}} , \frac{\partial \psi}{\partial x_{j}} \right] \]

Let \( v_{1}, v_{2} \) be a basis of 2-dim \( V \)

\( v_{1}^{*}, v_{2}^{*} \) the dual basis

\( (v_{1}^{*} \cdot v_{2}^{*})(\sum_{i} w_{i} \cdot v_{i}, \sum_{i} u_{i} \cdot v_{i}) = w_{1}u_{2} - w_{2}u_{1} \)

\( [v_{1} \cdot u_{2}](\sum_{i} w_{i} \cdot v_{i}, \sum_{i} u_{i} \cdot v_{i}) = w_{1}u_{2} - w_{2}u_{1} \)}
Differentiating one-forms

Let \( \eta = \gamma_1 \, d\phi^1 + \gamma_2 \, d\phi^2 \)

be a 1-form. We define a new surface.

We define

\[
\eta_i(s) = \left( \frac{\partial \eta_2}{\partial x^i} (\varphi^{-1}(s)) - \frac{\partial \eta_1}{\partial x^i} (\varphi^{-1}(s)) \right) \, d\phi^1 \wedge d\phi^2
\]

a 2-form.

I claim this doesn't change if we change parameters \( \varphi : u \to s \)

\( \tilde{\varphi} : v \to s \)

Statement

\[
\eta = \tilde{\gamma}_1 \, d\tilde{x}^1 + \tilde{\gamma}_2 \, d\tilde{x}^2
\]

Now

\[
\frac{\partial \psi}{\partial x^i} (\varphi^{-1}(s)) = \sum_j \frac{\partial (X^j \circ \varphi)}{\partial x^i} (\varphi^{-1}(s)) \frac{\partial x^j}{\partial x^i}
\]

\[
\eta_i(s) = \sum \tilde{\eta}_j(s) \frac{\partial (X^j \circ \tilde{\varphi})}{\partial x^i} (\varphi^{-1}(\tilde{\varphi}))
\]

already done
and \( d\hat{X}^j(s) = \sum \frac{\partial (X-^o\psi)^j}{\partial X^i} \delta \hat{y}^i(s) \)

(Follows by duality from relation for tangent vectors.)

then

\[
\frac{\partial \eta^o \psi}{\partial x^j} (\psi^{-1}s) = \sum \frac{\partial}{\partial x^j} \left( \tilde{\eta}^m \psi \frac{\partial (X-^o\psi)^m}{\partial x^i} \right)
\]

\[
= \sum \frac{\partial \tilde{\eta}^m \psi}{\partial x^j} (X^{-1}s) \frac{\partial (X-^o\psi)^m}{\partial x^j} (x) + \sum (\tilde{\eta}^m \psi) \left( \psi^{-1} \right) \frac{\partial^2 (X-^o\psi)^m}{\partial x^j \partial x^i} (x)
\]