Lecture 25

In practice we don't use partitions of unity to calculate integrals. Usually do something like:

Find a parametrisation $\varphi : U \to S$ such that $S - \varphi(U)$ is of "measure zero". or lower dim^{l} than S.

$$\varphi : \mathbb{R}^2 \longrightarrow S^2 \qquad \cancel{\varphi(\mathbb{R}^2) \subseteq C}$$

$$S^2 - \varphi(\mathbb{R}^2) = \{(1,0,0)\}$$

Then
$$\int_S \omega = \int_{\varphi(U)} \omega = I_\varphi(\omega)$$

6.3 Volume forms

Lemma 6.14

If V is a vector space with an inner product & $e^1 .. e^n$, $f^1 .. f^n$ o/e then

$$[e^1, \dots, e^n] = \pm [f^1, \dots, f^n]$$

Proof $f^i = \sum X_{ij} e^j$ $\quad [e^1 .. e^n] = \det X \, (f^1 .. f^n]$ (Corollary 6.6)

o/e $\Rightarrow$ $X X^t = I$

$\therefore (\det X)^2 = 1 \quad \therefore \det X = \pm 1.$ //

Def^n 6.15 If $S \subseteq \mathbb{R}^N$ is oriented and $e^1 \dots e^n$ is o/e wrt the inner product on $\mathbb{R}^N$ restricted to $T_s S$, and $(e^1, \dots e^n]$ is positive for the orientation we call $[e^1, \dots, e^n]$ the volume form at s. ~~If ... is oriented~~ Denote vol_S

Clearly $(e^1 \dots e^n]$ is independent of choices by ~~...~~ Prop^n 6.6 & 6.7

Prop^n 6.16 If $\varphi : U \to S$ is a parametrisation & $g_{ij} = \langle \frac{\partial \varphi}{\partial x^i}, \frac{\partial \varphi}{\partial x^j} \rangle$

then

$$\mathrm{vol}_S = \sqrt{\det(g_{ij})} \left[\frac{\partial \varphi}{\partial x^1}, \dots, \frac{\partial \varphi}{\partial x^n} \right]$$

Proof

Choose an o/e basis $e^1 \dots e^n$ & set

$$\frac{\partial \varphi}{\partial x^j} = \sum_{i=1}^{n} X_{ji} e^i$$

$$g_{ij} = \left\langle \frac{\partial \varphi}{\partial x^i}, \frac{\partial \varphi}{\partial x^j} \right\rangle = \sum X_{il} X_{jk} \langle e^l, e^k \rangle$$

$$= \sum X_{il} X_{jl}$$

$$= \sum X_{il} X^t_{lj}$$

$$\therefore \quad \det g_{ij} = (\det X)^2$$

$\underline{\text{, vam 6.6}}$ E

$$vol_S = [e^1, \dots, e^n] = \det X \left[\frac{\partial \varphi}{\partial x^1}, \dots, \frac{\partial \varphi}{\partial x^n}\right] //$$

Prop 6.17

If $\Sigma \subseteq \mathbb{R}^3$ is a surface & n the unit normal then

$$vol_\Sigma (v, w) = \langle v \times w, n \rangle$$

Proof Choose e^1, e^2 o/e $n = e^1 \times e^2$

Let $v = v_1 e^1 + v_2 e^2$, $w = w_1 e^1 + w_2 e^2$

then
$$\langle v \times w, n \rangle = v_1 w_2 - v_2 w_1$$

& $vd_\Sigma(v, w) = \det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} = v_1 w_2 - v_2 w$

(Prop 6.16)

<u>Corollary 6.18</u> As above then

$$R\, vd_\Sigma(v, w) = \langle \pi(v) \times \pi(w), n \rangle$$

<u>Proof</u> Take $v = e^1$ $w = e^2$ ~~then~~ $\pi e^i = \sum \pi^i{}_j e^j$

$LHS = R$, $RHS = \langle \det(\pi^i{}_j)\, e_1 \times e_2, n \rangle$

$$= \det \pi^i{}_j = R$$

By multilinearity both sides agree on any two vectors

Example Surface area of S^2_r

Choose usual $\psi : (0, 2\pi) \times (0, \pi) \to S^2_r$

$S^2_R - \psi((0,2\pi) \times (0,\pi))$ has measure zero

$$\int_{S^2_R} \mathrm{vol} = \int_{\mathcal{U}} \mathrm{vol}\left(\frac{\partial \psi}{\partial \phi}, \frac{\partial \psi}{\partial \theta}\right) d\phi \, d\theta$$

$$\frac{\partial \psi}{\partial \phi} \times \frac{\partial \psi}{\partial \theta} = R^2 \sin \phi \; n \qquad \text{(Lec 21)}$$

$$\mathrm{vol}\left(\frac{\partial \psi}{\partial \phi}, \frac{\partial \psi}{\partial \theta}\right) = R^2 \sin \phi$$

$$\int \mathrm{vol} = \int_0^{\pi} \int_0^{2\pi} R^2 \sin \phi \, d\theta \, d\phi$$

$$= 4\pi r^2$$

We want to compute $\frac{1}{2\pi} \int_{\Sigma} R \, \mathrm{vol}_{\Sigma}$

for example. $S^2_r \sim R = \frac{1}{r^2}$ $\therefore$

$$\boxed{\frac{1}{2\pi} \int_{S^2_r} R \, \mathrm{vol}_{S^2_r} = 2.}$$

More generally we want to show that

$$\frac{1}{2\pi} \int_{\Sigma} R \, v d_{\Sigma}$$

~~assumptions~~ doesn't change if we reform Σ!

Need a few new ideas.

6.4 One-form

If V is a vector space a one-form is a linear map $V \to \mathbb{R}$. The set of ~~Recall~~ all we denote by V^*. Recall that if $v_1, \ldots, v_n$ is a basis ~~we can~~ of V we can define $v_i^* \in V^*$ by

$$v_i^* \left(\sum \alpha_j v_j \right) = \alpha_i.$$

Then $v_1^*, \ldots, v_n^*$ is a basis of V^* called the dual basis of V^*.

(Ex – show $v_1^* \ldots v_n^*$ is a basis)

Note $\boxed{v_i^*(v_j) = \delta_{ij}}$.

If V is 2-dim^{l} and $w_1, w_2 \in V^*$ we define their <u>wedge product</u> by

$(w_1 \wedge w_2) \in (\det V)^*$ by

$$(w_1 \wedge w_2)(v_1, v_2) = w_1(v_1)w(v_2) - \cancel{w_2(v_1)w_1(v_2)}\ w_1(v_2)w_2(v_1)$$

<u>Ex</u> $w_1 \wedge w_2$ is multilinear & ~~antisymm~~ antisymmetric.

↓ End lec 25

~~If S is a submanifold then there~~

~~Let $S \subseteq \mathbb{R}^N$ be a submanifold.~~

~~Defn 6.16 A choice of a linear map $T_sS \to \mathbb{R}$ for all s is a~~

If $S \subseteq \mathbb{R}^N$ is a submanifold & $f : S \to \mathbb{R}$ is smooth then $f'(s) : T_sS \to \mathbb{R}$ is a 1-form. From now on call it df or $df(s)$, the exterior derivative of f.

Let $\psi : U \to S$ be a parametrisation. Then $\psi^{-1} : \psi(U) \to \mathbb{R}^n$ & ~~ψ^{-1}~~ $(\hat{\psi}^1, \dots$

Denote it by $\hat{\psi} = \psi^{-1}$ and let $\hat{\psi} = (\hat{\psi}^1, \dots, \hat{\psi}^n)$