In practice we don't partition a unity to calculate integrals. Usually do something like:

Find a parameterization \( Y: U \to S \)
such that \( S - (Y(U)) \) is of "measure zero". For example, unit circle than \( S \).

\[
Y: \mathbb{R}^2 \to S^2
\]

\[
S^2 - Y(\mathbb{R}^2) = \{ (1,0,0) \}
\]

Then
\[
\int_S w = \int_{Y(U)} w = \int_{Y(U)} Y^\top w
\]
63 Volume form

**Lemma 6.104**

If \( V \) is a vector space with an inner product \( \langle e_1, \ldots, e_n \rangle \), \( f'_1 \ldots f'_n \) of the form \( \langle e_1, \ldots, e_n \rangle = \pm \langle f'_1, \ldots, f'_n \rangle \)

**Proof**

\[
\langle f'_1, \ldots, f'_n \rangle = \sum_{ij} x_{ij} e_i \otimes e_j \] \[
\langle e_1, \ldots, e_n \rangle = \text{det}(x) \langle f'_1, \ldots, f'_n \rangle
\]

**Corollary 6.6**

\[
x x^t = I \]

\[
\text{det}(x) = \pm 1
\]

**Definition 6.185**

If \( S \subseteq \mathbb{R}^n \) and \( e'_1 \ldots e'_n \) is an orthonormal basis for \( S \), we call \( \langle e'_1, \ldots, e'_n \rangle \) the *volume form* at \( S \).

Clearly \( \langle e'_1, \ldots, e'_n \rangle \) is independent of choice by Prop. 6.6 & 6.7.

**Prop. 6.186**

If \( U \rightarrow S \) is a parametrization \( \frac{\partial y}{\partial x} = \begin{pmatrix} \frac{\partial y}{\partial x_1} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{pmatrix} \)

then

\[
\text{vol}_S = \sqrt{\text{det} \left( g_{ij} \right)} \left[ \frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n} \right]
\]
Proof

Choose an o/l basis $e^1, ..., e^n$ and let

$$\frac{\partial \psi}{\partial x^j} = \sum_{i=1}^n X_{ij} e^i$$

$$g_{ij} = \left< \frac{\partial \psi}{\partial x^i}, \frac{\partial \psi}{\partial x^j} \right> = \sum_{k=1}^n X_{ik} X_{jk} \left< e^i, e^k \right>$$

$$= \sum_{k=1}^n X_{ik} X_{jk}$$

$$= \sum_{k=1}^n X_{ik} X_{kj}$$

$$\therefore \text{det } g_{ij} = (\text{det } X)^2$$

Prop 6.17

If $\Sigma \subseteq \mathbb{R}^3$ is a surface and $n$ the unit normal, then

$$\text{vol}_\Sigma (v, w) = \left< v \times w, n \right>$$

Proof

Choose $e^1, e^2$ o/l $n = e^1 \times e^2$

Let $v = v_1 e^1 + v_2 e^2$, $w = w_1 e^1 + w_2 e^2$
Then
\[ \langle vxw, n \rangle = v_1 w_2 - v_2 w_1 \]

& \[ nd_x v (v, w) = \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} = v_1 w_2 - v_2 w_1 \quad \text{(Prop 6.16)} \]

**Corollary 6.18** As above then

\[ R v \delta_x (v, w) = \langle \pi(v) \times \pi(w), n \rangle \]

**Proof** Take \( v = e^i \), \( w = e^j \) then \( \pi e^i = \sum \pi^{ij} e^j \)

\[ \text{LHS} = R, \quad \text{RHS} = \langle \det (\pi^{ij}) e_1 \times e_2, n \rangle \]

\[ = \det (\pi^{ij}) = R \]

By multilinearity both sides agree on any two vectors.
Example  Surface area of $S^2_r$

Choose usual $\gamma : (0, 2\pi) \times (0, \pi) \to S^2_r$

$\delta^{2}_r = \gamma((0, 2\pi) \times (0, \pi))$ has measure $2\pi r$

\[
\int v d\ell = \int v d\ell \left(\frac{\partial y}{\partial \theta}, \frac{\partial y}{\partial \phi}\right) d\phi d\theta
\]

$\frac{\partial y}{\partial \theta} \times \frac{\partial y}{\partial \phi} = r^2 \sin \phi \quad \nu = (0, 0, 1) \quad (\text{Lec 21})$

$vd(\frac{\partial y}{\partial \theta}, \frac{\partial y}{\partial \phi}) = r^2 \sin \phi$

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\pi} r^2 \sin \phi \quad d\phi d\theta
\]

$= 4\pi r^2$

We want to compute $\frac{1}{2\pi} \int RvdS_r$

For example, $S^2_r \sim R = \frac{1}{r^2}$

$\frac{1}{2\pi} \int_{S^2_r} RvdS_r = 2$. 

\[
\]
More generally we want to show that
\[ \frac{1}{2\pi^2} \int_R V d\xi \]
doesn't change if we re-form \( \xi \),
need a few new ideas.

6.4 One-form

If \( V \) is a vector space a one-form is a linear map \( V \rightarrow \mathbb{R} \). The set of all we denote by \( V^* \). Recall that if \( \xi_1, \ldots, \xi_n \) is a basis of \( V \),
we can define \( \xi^* \in V^* \) by
\[ \xi^*(\sum a_i \xi_i) = a_i \]
Then \( \xi_1^*, \ldots, \xi_n^* \) is a basis of \( V^* \)
called the dual basis of \( V \).

Note \[ \xi^*(\xi_j) = \delta_{ij} \]