# Lecture 24

<u>Recall</u> $\det(V^*)$ is all

multilinear, antisymmetric maps

$$\omega : \underbrace{V \times \dots \times V}_{n = \dim V} \longrightarrow \mathbb{R}$$

~~We just proved~~

~~If $\omega \in$~~

If $v^1 \dots v^n$ basis of $V$ then define

$$[v^1, \dots, v^n](w^1, \dots, w^n) = \det(X^i{}_j)$$

where $$w^i = \sum_{j=1}^{n} X^i{}_j \, v^j$$

Then if $\omega \in \det(V^*)$

$$\omega = \omega(v^1, \dots, v^n)\,[v^1, \dots, v^n]$$

$\therefore$ $\det(V^*)$ is 1-dim$^{e}$

<u>Example of proof</u> $\dim V = 2$. $v^1, v^2$ basis

$w^1$ ~~$\rightarrow w$~~ $w^2 \in V$

$$w^i = X^i{}_1 v^1 + X^i{}_2 v^j$$

$$\omega(w^1, w^2) = \omega(X^1{}_1 v^1 + X^1{}_2 v^2, \; X^2{}_1 v^1 + X^2{}_2 v^2)$$

$$= x^1_1 x^2_1 \underbrace{\omega(v^1, v^1)}_{=0} + x^1_1 x^2_2\, \omega(v^1, v^2)$$

$$+ x^1_2 x^2_1 \underbrace{\omega(v^2, v^1)}_{= -\omega(v^1, v^2)} + x^1_2 x^2_2 \underbrace{\omega(v^2, v^2)}_{=0}$$

$$= \omega(v^1, v^2)\left( x^1_1 x^2_2 - x^1_2 x^2_1 \right)$$

$$= \omega(v^1, v^2) \det(X)$$

$$= \omega(v^1, v^2)\, [v^1, v^2](w^1, w^2).$$

## Corollary 6.6

If $w^1, \dots w^n$, $v^1, \dots v^n$ are two bases

& $w^i = \sum_j X^i_j v^j$ then

$$[w^1, \dots, w^n] = \det(X^i_j)\, [v^1, \dots, v^n]$$

Proof

( $\omega = \omega(w^1, \dots, w^n)\, [w^1 \dots w^n]$

~~$[w^1, \dots, w^n] = [w^1 \dots w^n]$~~

$[v^1, \dots, v^n] = [v^1, \dots, v^n](w^1, \dots, w^n)\, [w^1, \dots, w^n]$

$= \det(X^i_j)\, [w^1, \dots, w^n]$ )

If $S \subseteq \mathbb{R}^N$ is a $n$ dim$^l$ submanifold then we say it is <u>oriented</u> if we have chosen continuously one half of $\det(T_s S^*) - \{0\}$. Call an $n$-form in this half positive.

If $\psi : U \to S$ is a parametrisation call it oriented if $\left[\frac{\partial \psi}{\partial x^1}, \dots, \frac{\partial \psi}{\partial x^n}\right]$ is positive.

<u>Prop$^n$ 6.7</u>

Let $\chi : V \to S$, $\psi : U \to S$ be oriented parametrisation with $\chi(V) = \psi(U)$ and let $\rho = \psi^{-1} \circ \chi : V \to U$. Then

$$\det J(\rho) > 0.$$

<u>Proof</u>

$$\chi = \psi \circ \rho$$

$$\frac{\partial \chi}{\partial x^i}(x) = \sum \frac{\partial \psi}{\partial x^j}(\rho(x)) \frac{\partial \rho^j}{\partial x^i}(x)$$

$$= \sum \underbrace{\frac{\partial \rho^j}{\partial x^i}(x)}_{J(\rho)^j_i} \frac{\partial \psi}{\partial x^j}(\rho(x))$$

So from Corollary 6.6

$$\left[\frac{\partial \psi}{\partial x^1}(\rho(x)) \;\ldots\; \frac{\partial \psi}{\partial x^n}(\rho(x))\right]$$

$$= \det J(\rho) \left[\frac{\partial \chi}{\partial x^1}, \ldots, \frac{\partial \chi}{\partial x^n}\right].$$

+ve multiples of each other $\therefore$ $\det J(\rho) > 0$ //

A $n$-form $\omega$ on $S$ is a choice of $n$ form at each $s \in S$. If $\psi : U \to S$ we have

$$\omega = \omega\left(\frac{\partial \psi}{\partial x^1}, \ldots, \frac{\partial \psi}{\partial x^n}\right)\left[\frac{\partial \psi}{\partial x^1}, \ldots, \frac{\partial \psi}{\partial x^n}\right]$$

$$\underbrace{\omega\left(\frac{\partial \psi}{\partial x^1}, \ldots, \frac{\partial \psi}{\partial x^n}\right)}_{= \omega_\psi} \qquad \omega_\psi : U \to \mathbb{R}$$

Call $\omega$ smooth if we can cover $S$ by parametrisations so that each $\omega_\chi$ is smooth.

~~From Proof of~~ 6. we have

<u>Prop^n 6.8</u> $\chi, \psi$ as above then

$$\omega_\chi = \det J(\rho) \cdot \omega_\psi \circ \rho$$

<u>Proof</u> $\cancel{\omega_\chi} =$

We have

~~w(s) =~~

$$w(\chi(x)) = w_\chi(x) \left[\frac{\partial \chi}{\partial x^1}(x), \ldots, \frac{\partial \chi}{\partial x^n}(x)\right]$$

$$= w_\chi(x) \det J(\rho)(x)^{-1} \left[\frac{\partial \psi}{\partial x^1}(\rho(x)), \ldots \frac{\partial \psi}{\partial x^n}(\rho(x))\right]$$

& $w(\chi(x)) = w(\psi(\rho(x))) =$

$$= w_\psi(\rho(x)) \left[\frac{\partial \psi}{\partial x^1}(\rho(x)) \ldots \frac{\partial \psi}{\partial x^n}(\rho(x))\right]$$

$$\therefore \quad w_\chi(x) = \cancel{\text{that}} \det J(\rho)(x)\, w_\psi(\rho(x)) \quad //$$

This shows that being ~~as~~ a smooth n-form doesn't depend on the choice of parametrisation as $w_\chi$ is smooth $\iff w_\psi$ is smooth.

~~If ω is~~

**Def$^n$ 6.9** If $\omega$ is a smooth $n$-form on $S$ we define

$$\operatorname{supp}(\omega) = \{ x \mid \omega(x) \neq 0 \}$$

**Def$^n$ 6.10** If $\psi : U \to S$ is an oriented param & $\omega$ an $n$-form with $\operatorname{supp}\omega \subseteq U$ then we define

$$I_\psi(\omega) = \int_U \omega_\psi \, dx^1 \dots dx^n = \int_U \omega\left(\frac{\partial \psi}{\partial x^1}(x), \dots \frac{\partial \psi}{\partial x^n}(x)\right) dx^1 \dots dx^n$$

**Prop$^n$ 6.11** If $\chi : V \to S$, $\psi : U \to S$ oriented, with $\chi(V) = \psi(U)$

then

$$I_\chi(\omega) = I_\psi(\omega)$$

$$\begin{tikzcd} V \arrow[r, "\chi"] \arrow[d, "\rho"'] & S \\ U \arrow[ur, "\psi"'] & \end{tikzcd}$$

**Proof**

$$I_\psi(\omega) = \int_U (\omega)_\psi \, dx^1 \dots dx^n$$

$$= \int_V \det(J(\rho)) \, \omega_\psi \circ \rho \, dx^1 \dots dx^n$$

$$= \int_V \omega_\chi \, dx^1 \dots dx^n$$

$$= I_\chi(\omega)$$

NB $\det J(\rho) > 0$ as $\chi, \psi$ both oriented.
$\Rightarrow \omega_\psi, \omega_\chi$ +ve multiples of each other

To be sure the integral exist we should insist that supp $\omega$ is compact.

~~In all our examples where we can~~

What about forms that don't have support ~~on~~ in the image of some parametrisation? We use a technical trick called a partition of unity.

Def 6.12

If $S$ ~~has~~ is covered by open sets $W_1, \dots, W_k$ with a partition of unity subordinate to the cover. Is a collection of maps $\rho_i : S \to \mathbb{R}$ s.t.

1. $\sum \rho_i = 1$
2. $\rho_i \geq 0$
3. $\operatorname{supp} \rho_i \subseteq W_i$

Note the $\rho_i \cdot \omega$ is an $n$-form with

$$\operatorname{supp}(\rho_i \omega) \subseteq \operatorname{supp}(\rho_i) \subseteq W_i$$

& $$\omega = \sum (\rho_i \omega)$$

~~In other words a partition of~~

If $\varphi_i : U_i \to S$ is a collection of parametrisations with $\varphi_i(U_i) = W_i$ then we define

$$\int_S \omega = \sum_i I_{\varphi_i}(\rho_i \omega)$$

Prop$^n$ 6.13 The integral of an $n$-form is well-defined independent of choice of parametrisations and partition of unity.

End of lecture