Lecture 23

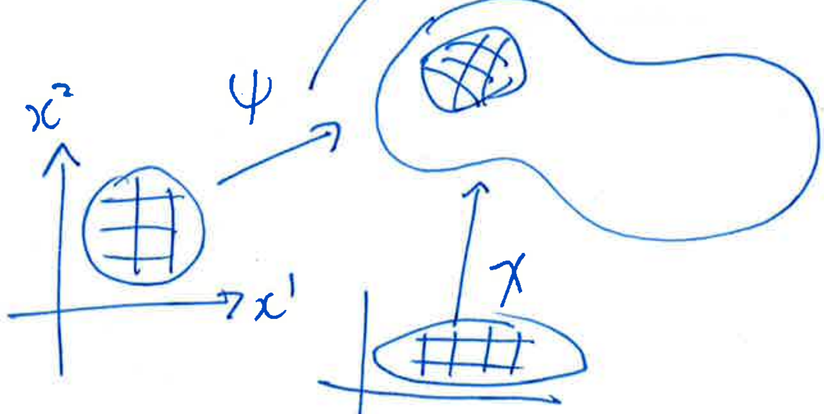
§6 Integration

Motivation: want to integrate $\int_{\Sigma} R\,dA$

Turns out we need to think of "$R\,dA$" as a 2-form and integrate the 2-form.

~~# p-forms on a manifold * can be integrated over p dim' submanifold~~

To motivate this think about integrating ~~of~~ in the image of a parametrisation using parameters. Need to integrate in $\mathbb{R}^n$ and see how it changes when we change parametrisation.

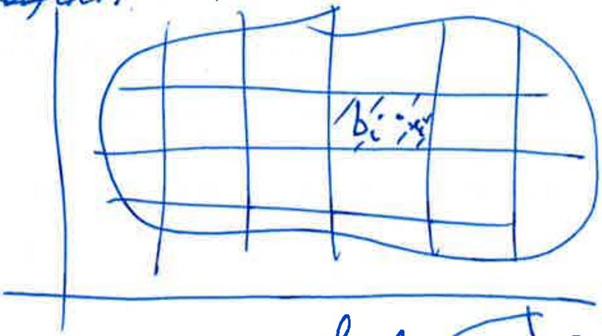


$\int f \circ \varphi \, dx^1 dx^2$

If R = Gaussian curvature

Lecture 23

we want $\int_S R$ ~~surf~~ dA. How to do this?



$\lim \sum f_i(x_i) \operatorname{vol}(b_i)$

§6 Integration

6.1 Integration in $\mathbb{R}^n$

Recall

~~We are particularly interested~~ in

Let R be a closed bounded region in $\mathbb{R}^n$.

& $f: \cancel{\mathbb{R}} R \to \mathbb{R}$ cts. ~~Define~~ We define

$\int_R f \, dx^1 \ldots dx^n$. It satisfies:

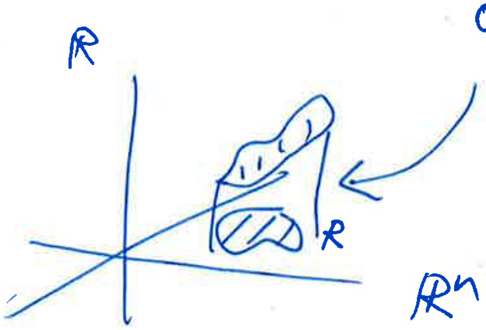
Prop^n 6.1

1. $f \longmapsto \int_R f \, dx^1 \ldots dx^n$ is linear

2. If $f(x) \geq 0 \ \forall x \in R \Rightarrow \int_R f \, dx^1 \ldots dx^n \geq 0$

3. If $f(x) \geq 0$ $\int_R f \, dx^1 \ldots dx^n$ is the volume of

$$\left\{ (x^1, \ldots, x^{n+1}) \;\middle|\; \begin{array}{l} (x^1, \ldots, x^n) \in R \\ 0 \leq x^{n+1} \leq f(x^1 \ldots x^n) \end{array} \right\}$$



(4) If R_1, R_2 are disjoint & $R = R_1 \cup R_2$

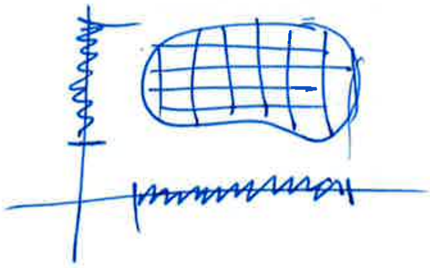
$$\int_R f\, dx^1 \ldots dx^n = \int_{R_1} f\, dx^1 \ldots dx^n + \int_{R_2} f\, dx^1 \ldots dx^n$$

How do we calculate? . Do $\mathbb{R}^2$ case

Thm 6.2 (Fubini) If $R \subseteq \mathbb{R}^2$, closed, bdd
$f : R \to \mathbb{R}$ cts

$$\int_R f\, dx\, dy$$

$$= \int \left(\int f(x,y)\, dx \right) dy$$



$$= \int \left(\int f(x,y)\, dy \right) dx$$

Def 6.3 If $U \subseteq \mathbb{R}^n$ open & $f : U \to \mathbb{R}$
cts define the support of f

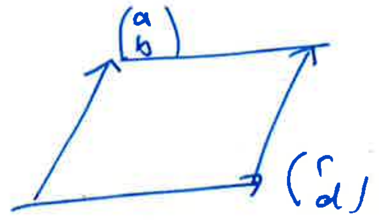
$$\text{supp}(f) = \overline{\{ x \mid f(x) \neq 0 \}}$$

= smallest closed set containing
$\{ x \mid f(x) \neq 0 \}$

If $\left[\begin{matrix} U \subseteq_{\text{open}} \mathbb{R}^n \\ f : U \to \mathbb{R} \end{matrix} \right.$ & $\text{supp}(f)$ is bounded

Define $\int_U f\, dx^1 \ldots dx^n = \int_{\text{supp}(f)} f\, dx^1 \ldots dx^n$

Recall that if



is a parallelogram then its area

$$= \left|\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}\right|$$

More generally if $X = (v_1\ v_2 \dots v_n)$

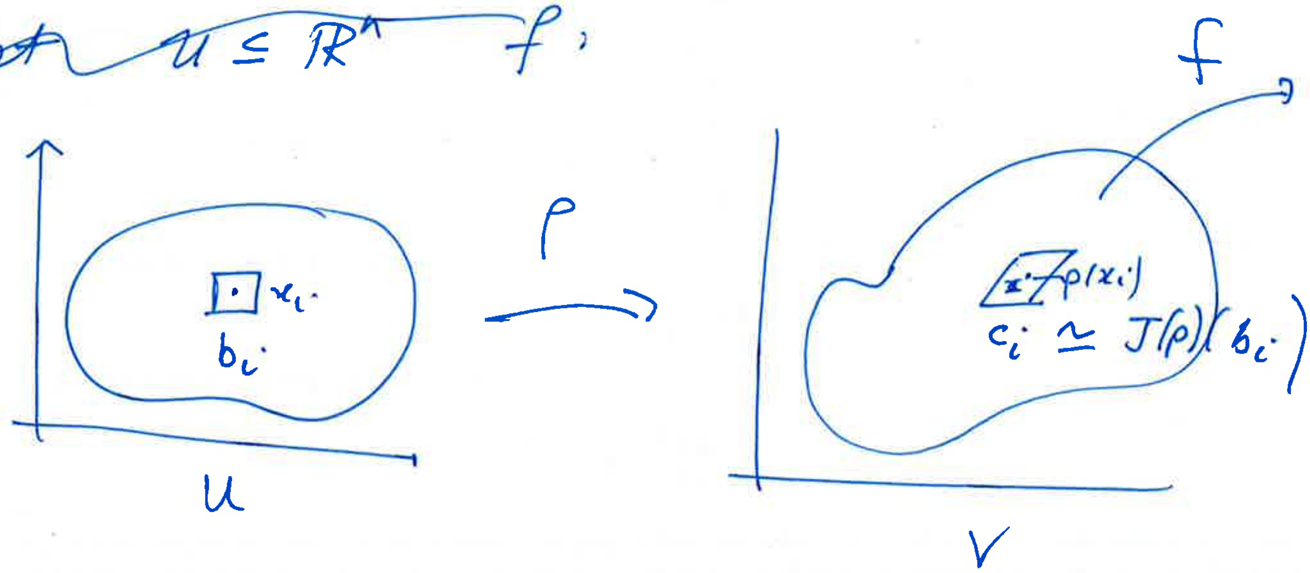
vol ~~Area~~ (~~convex hull $v_1, \dots v_n$~~ $\sum t_i v_i \quad 0 \le t_i \le 1$)

$= \det X.$

Even more generally if $D \subseteq \mathbb{R}^n$

$$\mathrm{vol}(X(D)) = |\det X| (\mathrm{vol}\, D).$$

~~Let $U \subseteq \mathbb{R}^n$ f:~~



$$\int_V f \sim \sum_i f(\rho(x_i))\, \mathrm{vol}(c_i) \sim \sum f(\rho(x_i)) |\det J(\rho)|\, \mathrm{vol}(b_i)$$

$$\sim \int_U f \circ \rho\, |\det J(\rho)|\, dx^1 \dots dx^n$$

Th^m 6.4 (Change of variable)

Let $U \underset{open}{\subseteq} \mathbb{R}^n$, $f : U \to \mathbb{R}$ cts & $\varphi : U \to V$ a diffeo^m for $V \underset{open}{\subseteq} \mathbb{R}^n$.

Let $f : V \to \mathbb{R}$ have bounded support then

$$\int_V f \, dx^1 \ldots dx^n = \int_U f \circ \varphi \, |\det(J(\varphi))| \, dx^1 \ldots dx^n$$

where

$$J(\varphi) = \begin{bmatrix} \frac{\partial \varphi^1}{\partial x^1} & \cdots & \frac{\partial \varphi^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial \varphi^n}{\partial x^1} & \cdots & \frac{\partial \varphi^n}{\partial x^n} \end{bmatrix}$$

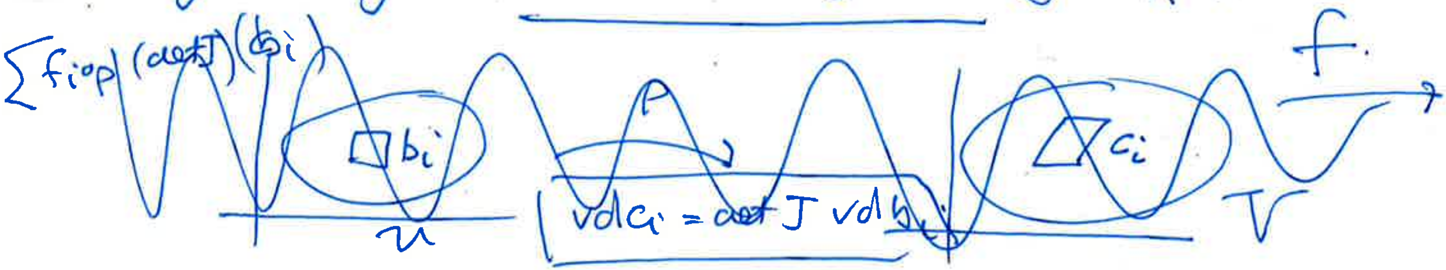
~~Mention how linear transform changes vol.~~

(Note the absolute value sign here)

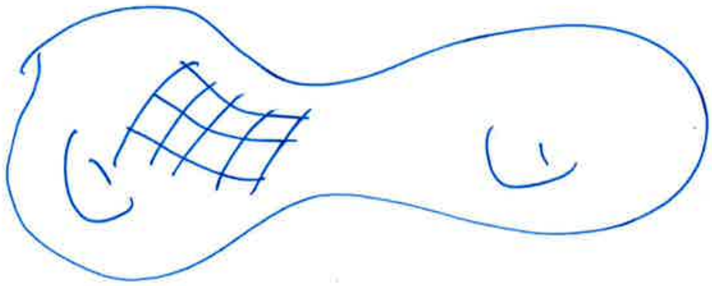
To integrate over submanifolds we need to find a way of integrating which is independent of parameter. DRAW PICTURE

This amounts to including the $dx^1 \ldots dx^n$ into the integrand and integrating volume forms

$\sum f_i \, \mathrm{vol}(c_i)$



Alternatively you could imagine trying to divide the submanifold into "boxes".



Then what we integrate is something that measures the size of the box like $f(x)\,dx^1 \ldots dx^n \sim f(x_i)\,\mathrm{vol}(b_i)$

We go back to the idea that the size of a box with sides $b_1, \ldots, b_n$ in $\mathbb{R}^n$ is

$|\det(b_1\ b_2 \ldots b_n)|$. Consider $\det(b_1, \ldots, b_n)$

Think of this as a function of n vectors $b_1, \ldots, b_n$. It has various properties we capture as follows.

~~On a surface submanifold can~~ 23/7

6.2 Volume form and integration

Let V be a vector space of ~~degree~~ dim^n n. An <u>n-form</u> is a ~~linear~~ map

$$\omega : \underbrace{V \times \cdots \times V}_{n} \longrightarrow \mathbb{R}$$

which is multilinear & antisymmetric

<u>Multilinear means:</u>

$$\omega(v_1, \ldots, v_{i-1}, au + bw, v_{i+1}, \ldots, v_n)$$

$$= a\,\omega(v_1 \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_n) + b\,\omega(v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_n)$$

$\forall v_j, u, w \quad \forall a, b \quad \forall i.$

<u>antisymmetric means</u>

$$\omega(v_1, \ldots, v_i, v_{i+1}, \ldots, v_n) = -\,\omega(v_1, \ldots, v_{i+1}, v_i, \ldots v_n)$$

~~[illegible]~~

In particular $\omega(v_1 \ldots v, \ldots v, \ldots v_n) = 0$.

<u>Examples</u> (1) If $V = \mathbb{R}^n$

$\omega(v_1 \ldots, v_n) = \det(v_i^{\,j})$ is ~~anti~~ an n-form.

(2) If $v^1, \dots, v^n$ is a basis of V define $[v^1, \dots, v^n](w^1, \dots, w^n)$

$$= \det(X^i_j)$$

where $w^i = \sum X^i_j \, v^i$

~~then~~ (3) If $\Sigma \subseteq \mathbb{R}^3$ is an oriented surface with normal n at $s \in \Sigma$ then

$$\omega(V, w) = \langle \cancel{V \wedge W}\, V \times W, n \rangle$$

is a 2-form on $T_s \Sigma$.

~~Prop^n 6.5~~

Denote by $\det(V^*)$ the vector space (check!) of all n-forms on V.

<u>Prop^n 6.5</u> The space of all n-forms is 1-dim^e. If $v^1, \dots, v^n$ is a basis of V & ω is an n-form then

$$\omega = \underbrace{\omega(v^1, \dots, v^n)}_{\text{number}} \underbrace{[v^1, \dots, v^n]}_{n\text{-form}}$$

<u>Proof</u> Let $w^i = \sum X_{ij} \, v_j$

then $\omega(w^1, \dots, w^n) = \omega\left(\sum X_{1j_1} v_{j_1}, \dots, \sum X_{nj_n} v_{j_n}\right)$

$$= \sum X_{1j_1} \dots X_{nj_n} \, \omega(v_{j_1}, \dots, v_{j_n})$$

But $w(v_{j_1},\dots,v_{j_n}) = \begin{cases} 0 & \text{if any two equal} \\ \operatorname{sgn}\begin{pmatrix} 1 & \dots & n \\ j_1 & \dots & j_n \end{pmatrix} w(v_1 \dots v_n) & \text{o/wise} \end{cases}$

standard properties of det tell us that.

$\therefore \quad \det(X_{ij}) = \sum_{\text{permutations } \pi} \operatorname{sgn}(\pi) X_{1\pi(1)} \dots X_{n\pi(n)}$

$\therefore \quad = \det(X_{ij})\, w(v^1, \dots, v^n)$

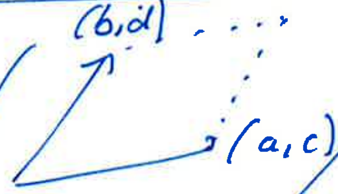
$= [v^1, \dots, v^n](w^1, \dots, w^n)\, \omega(v^1, \dots, v^n)$

End Lec 23

$$\therefore \quad \boxed{w = \omega(v^1, \dots, v^n)\, [v^1, \dots, v^n].}$$

~~Corollary 6.6 Let $w_1 \dots w_n$, $v_1 \dots v_n$ be bases & $w_i = \sum X_{ij} v_j$ then $[v_1 \dots v_n] = \det X_{ij}\, [w_1 \dots w_n]$~~

If $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $|\det(X)| = \text{area}\Big($ 

$\Big)$

More generally if $X = (v^1, \dots, v^n)$

$|\det(X)| = \text{vol}\{\sum t^i v^i \mid 0 \le t_i \le 1\}$

In many problems we want area to have a sign

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

This relates to orientation.