Recall $\Sigma \subseteq \mathbb{R}^3$ $\quad n$ normal $\quad \psi$ parameters

$$\alpha(v,w) = \sum \left\langle \frac{\partial^2 \psi}{\partial x^i \partial x^j}, n \right\rangle v^i v^j$$

$$v = \sum v^i \frac{\partial \psi}{\partial x^i}, \quad w = \sum w^i \frac{\partial \psi}{\partial x^i}$$

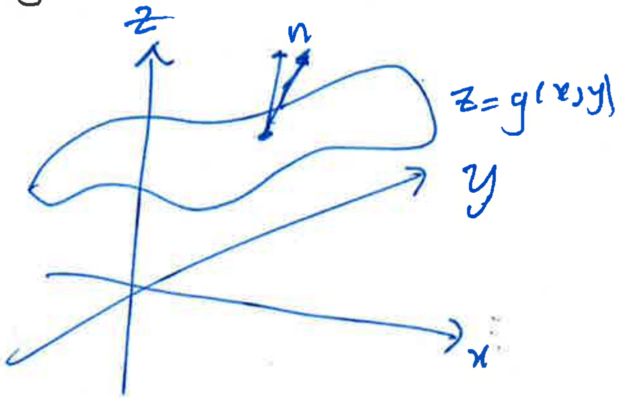
Waved sphere example.

Example $\qquad g : \mathbb{R}^2 \to \mathbb{R}$

$$\Sigma = \{ (x,y,z) \mid z = g(x,y) \}$$

graph of g

$$\psi(x,y) = (x, y, g(x,y))$$



$$\frac{\partial \psi}{\partial x} = \left(1, 0, \frac{\partial g}{\partial x}\right)$$

$$\frac{\partial \psi}{\partial y} = \left(0, 1, \frac{\partial g}{\partial y}\right)$$

$$\frac{\partial \psi}{\partial x} \times \frac{\partial \psi}{\partial y} = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)$$

$$\left\| \frac{\partial \psi}{\partial x} \times \frac{\partial \psi}{\partial y} \right\| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$$

$$n = \frac{\left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \left(0, 0, \frac{\partial^2 g}{\partial x^2}\right)^T, \ldots$$

$$\alpha\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial x}\right) = \left\langle \frac{\partial^2 \psi}{\partial x^2}, n \right\rangle = \frac{\frac{\partial^2 g}{\partial x^2}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

+ etc

recall $\alpha(v, w) = \langle -n'(s)(v), w \rangle$

we define

Defn 5.4 $\Pi : T_sS \to T_sS$

$$\Pi(v) = -n'(s)(v)$$

Assume v_1, v_2 o/e basis of T_sS. let

$$\Pi(v_i) = \sum \Pi_{ij} v_j$$

$$\alpha(v_i, v_j) = \langle \Pi(v_i), v_j \rangle = \Pi_{ij}$$

$\therefore \Pi_{ij}$ real, symmetric so has eigenvalue $\lambda_1 \leq \lambda_2$ & o/e eigenv v_1, v_2

22|3

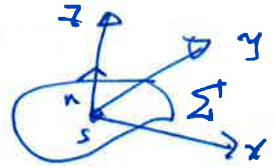
<u>Example</u> By translating & rotating

we can arrange $s = (0,0,0)$, $n = (0,0,1)$

Then $\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \in T_s n^{\perp} = x\text{-}y$ plane

Write Σ^+ locally as a graph so

$$g(0,0) = 0 \qquad \frac{\partial g}{\partial x}(0,0) = 0$$



$$\frac{\partial g}{\partial y}(0,0) = 0.$$ so

$$\frac{\partial \varphi}{\partial x}(0,0) = (1,0,0) \qquad \frac{\partial \varphi}{\partial y}(0,0) = (0,1,0)$$

↑ ok.

$z = x^1$ $y = x^2$.

$$\therefore \pi_{ij} = \cancel{\left[\frac{\partial^2 g}{\partial x^i \partial x^j}(0,0)\right]}$$

= ~~Hessian of g.~~

$$= \alpha\left(\frac{\partial \varphi}{\partial x^i}(0,0), \frac{\partial \varphi}{\partial x^j}(0,0)\right)$$

$$= \left[\frac{\partial^2 g}{\partial x^i \partial x^j}(0,0)\right]$$

Recall ~~Note~~ that we have

$\alpha(v, w) = (-n'(v), w)$ we define

~~Definition 5.4~~

$\Pi : T_s\Sigma \to T_s\Sigma$

$\Pi(v) = -n'_{(s)}(v)$.

With respect to an o/e basis $\{v_i\}$ let

$$\Pi(v_i) = \sum_{j=1}^{2} \Pi_{ij} v_j$$

then $\alpha(v_i, v_j) = (\Pi(v_i), v_j)$

$=$

$= \sum$

$= \sum (\Pi_{il})(v_l, v_j)$

$= \Pi_{ij}$

$\therefore$ Π_{ij} is symmetric (real)

so can be diagonalised wrt to an o/e basis

Talk about Hessian example

$\frac{\partial 4}{\partial x} = (1, 0, 0)$ $\quad \frac{\partial 4}{\partial y} = (0, 1, 0)$ o/e $\therefore$

$\Pi_{ij} = \frac{\partial^2 g}{\partial x^i \partial x^j}$

Def^n 5.5

(1) The eigenvalues λ_1, λ_2 of Π are called the principal curvatures.

(2) $\frac{1}{2} tr(\Pi) = \frac{1}{2}(\lambda_1 + \lambda_2)$ is called the mean curvature

(3) $K = \det(\pi) = \lambda_1 \lambda_2$ is called the Gaussian curvature. $\cancel{\left[G\text{-}B \quad \frac{1}{2\pi}\int K\right]}$

~~Not~~ If v has length 1 then & e_1, e_2 are an o/n basis of eigenvectors for π, $\lambda_1 \leq \lambda_2$. Let

$v = v_1 e_1 + v_2 e_2$ then

$\alpha(v,v) = \langle \pi(v), v \rangle$ = curvature in direction of v

$\lambda_1 \leq \lambda_2$

$= \lambda_1 v_1^2 + \lambda_2 v_2^2$.

If $\lambda_1 < \lambda_2$ then $\alpha(v,v)$

Minimum if $v_2 = 0$ $\quad v = e_1 \quad \alpha(v,v) = \lambda_1$

Maximum if $v_1 = 0$ $\quad v = e_2 \quad \alpha(v,v) = \lambda_2$

~~Recall $\alpha(v,v)$ = curvature in direction of~~

Example ① Cylinder

$\psi(\theta, t) = (R\cos\theta, R\sin\theta, t)$

$n = (\cos\theta, \sin\theta, 0)$.

$\frac{\partial n}{\partial t} = 0 = 0 \cdot \frac{\partial \psi}{\partial t}$

$\frac{\partial n}{\partial \theta} = (-\sin\theta, \cos\theta) = \frac{1}{R} \frac{\partial \psi}{\partial \theta}$

Principal curvat. $\quad 0, -\frac{1}{R}$

Mean curv $\quad 0 + \frac{1}{R} = \frac{1}{R}$

Gaussian curvature $\quad 0 \times \frac{-1}{R} = 0$

2216

~~(3) $\det(\Pi) = +\lambda_1\lambda_2$ is called the Gaussian curvature.~~

Example S^2

$$n = (\cos\theta \sin\phi,\ \sin\theta \sin\phi,\ \cos\phi) \quad \left(= \tfrac{1}{R}\psi\right)$$

$$n'\left(\frac{\partial \psi}{\partial \theta}\right) = \frac{\partial (n\circ\psi)}{\partial \theta} = \frac{1}{R}\frac{\partial \psi}{\partial \theta}$$

$$n'\left(\frac{\partial \psi}{\partial \phi}\right) = \frac{\partial (n\circ\psi)}{\partial \phi} = \frac{1}{R}\frac{\partial \psi}{\partial \phi}$$

$$\Pi\left(\frac{\partial \psi}{\partial \theta}\right) = -\frac{1}{R}\frac{\partial \psi}{\partial \theta}$$

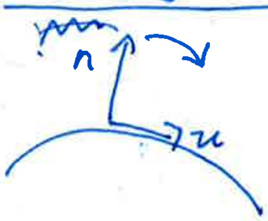
$$\Pi\left(\frac{\partial \psi}{\partial \phi}\right) = -\frac{1}{R}\frac{\partial \psi}{\partial \phi}$$

Principal curvatures $-\frac{1}{R}$, $-\frac{1}{R}$

Mean curvature $-\frac{1}{R}$

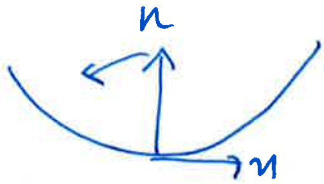
Gaussian curvature $+\frac{1}{R^2}$

Meaning of the sign



$n'(u) = \mu u \quad \mu > 0$

$\Pi(u) = \lambda u \quad \lambda < 0$



$$n'(u) = \mu u \qquad \mu < 0$$

$$\pi(u) = \lambda \mu \qquad \lambda > 0$$

$$(\lambda = -\mu).$$

Propⁿ 5.6

If $\{v_1, v_2\}$ is a basis for $T_s \Sigma$ and

$\alpha_{ij} = \alpha(v_i, v_j)$ & $g_{ij} = \langle v_i, v_j \rangle$

then $\boxed{K = \det(\pi_{ij}) = \det(\alpha_{ij}) / \det(g_{ij})}$

Proof

$\alpha(v, w) = (\pi(v), w)$

Let π have matrix

$$\pi(v_i) = \sum \pi_{ij} v_j.$$

Then

$$\alpha_{ij} = \alpha(v_i, v_j)$$

$$= \langle \pi(v_i), v_j \rangle$$

$$= \sum_l \langle \pi_{il} v_l, v_j \rangle$$

$$= \sum_l \pi_{il} g_{lj} \Rightarrow [\alpha_{ij}] = [\pi_{il}][g_{lj}]$$

$$\therefore \det \alpha_{ij} = \det \pi \det g_{ij}$$

//