Lecture 21: Friday 24th. September

* Any assignments give to me or put in box
* Not expecting you do all this.

Review (No need to repeat all of review partly for you!)

Recall that we defined for $\Sigma \subseteq \mathbb{R}^3$ a surface

$$\alpha : T_s\Sigma \times T_s\Sigma \longrightarrow \mathbb{R}$$

the second fundamental form on Σ as follows.

Choose a parametrisation $\psi : \mathcal{U} \longrightarrow \Sigma$, $\mathcal{U}$ open in $\mathbb{R}^2$, $\psi(0,0) = s$

Let $v, w \in T_s\Sigma$ $\quad v = \sum_{i=1}^{2} v^i \frac{\partial \psi}{\partial x^i}(0,0)$

$$w = \sum_{i=1}^{2} w^i \frac{\partial \psi}{\partial x^i}(0,0)$$

then

$$\alpha(v,w) = \sum_{i=1}^{2} \left\langle \frac{\partial^2 \psi}{\partial x^i \partial x^j}(0,0), n \right\rangle v^i w^j \quad \text{(*)}$$

where n is the unit normal at s.

We showed also that

$$\alpha(v,w) = -\langle n'(s)(v), w \rangle$$

~~and hence independent of ψ.~~

End of review

Prop^n 5.1 The second fundamental form $\alpha : T_s\Sigma' \times T_s\Sigma' \to \mathbb{R}$ satisfies

① α is bilinear and symmetric

i.e.
$$\begin{cases} \alpha(v,w) = \alpha(w,v) \\ \alpha(\lambda v+\mu u, w) = \lambda\alpha(v,w) + \mu\alpha(u,w) \\ \alpha(v, \lambda u+\mu w) = \lambda\alpha(v,u) + \mu\alpha(v,w) \end{cases}$$

② $\alpha(v,w) = -\langle n'(s)(v), w\rangle$

Proof

① Follows from formula for α — ⊛

② Proved last lecture.

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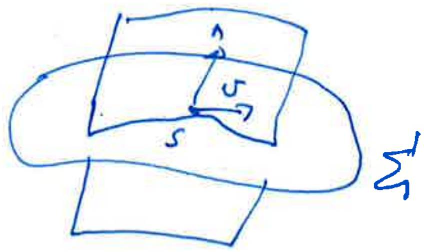
I want to relate α to curvature.

For this we choose $v \in T_s\Sigma_1$ and consider $(s+\text{span}\{v,n\}) \cap \Sigma_1$



I claim that near s this is a curve with curvature $\dfrac{|\alpha(v,v)|}{\|v\|^2}$.

First we need

Lemma 5.2 with notation as above $\exists\, \varepsilon > 0$ and a smooth map $a : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$\gamma(t) = s + tv + a(t)n \in \Sigma$$

for all $t \in (-\varepsilon, \varepsilon)$ & $a(0) = a'(0) = 0$.

Proof (Skip the proof if you want to.)

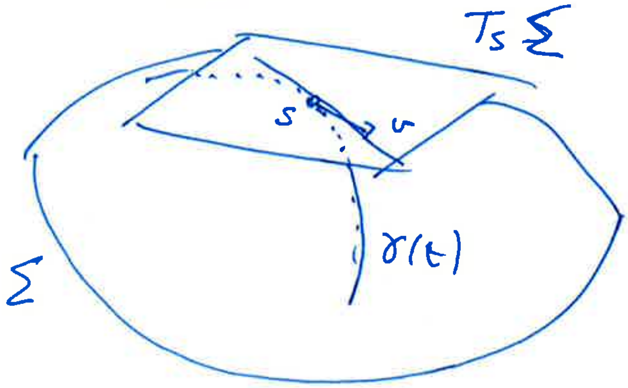
This is a trick we have done before. Choose local parameters $\varphi : U \to \Sigma$, $U \subseteq \mathbb{R}^2$, $\varphi(0) = s$ say.

Then define $F : U \times \mathbb{R} \longrightarrow \mathbb{R}^3$ (Picture on next page.)

$$F(x, y) = \varphi(x) + yn$$

Then $F(0,0) = \varphi(0) = s$. As we have seen before $F'(0,0)$ is invertible so locally F is a

Proof of Lemma 5.2



Consider $\tilde{\pi} : \mathbb{R}^3 \to T_s\Sigma$

$\tilde{\pi}(x) = (x-s) - \langle (x-s), n \rangle n$

$\perp$ projection on $T_s\Sigma$

Let $\pi = \tilde{\pi}|_S$.

Note $\pi(s) = 0$. Choose parameters $\varphi : U \to S$ $\varphi(0) = s$. Consider $\pi \circ \varphi$.

$(\pi \circ \varphi)'(0)(w) = \varphi'(0)(w)) - \langle \varphi'(0)(w), n \rangle n$

If $= 0$ then $\varphi'(0)(w) \propto n$ $\therefore$ $\varphi'(0)(w) = 0$ as $\varphi'(0)(w) \in T_s\Sigma$. But $\varphi'(0)$ is 1-1 $\therefore$ $\therefore$ $w = 0$. $\therefore$ $(\pi \circ \varphi)'(0)$ is 1-1 & invertible.

So by I F. Th locally $\pi \circ \varphi$ is invertible.

But $\pi^{-1} = \varphi \circ (\pi \circ \varphi)^{-1}$. So locally π is invertible. So for $|t| < \varepsilon$ we can define $\gamma(t) = \pi^{-1}(tv)$.

Then

$$tv = \pi(\gamma(t)) = \gamma(t) - s - \langle \gamma(t) - s, n \rangle n$$

$$\therefore \quad \gamma(t) = s + tv + \underbrace{\langle (\gamma(t) - s), n \rangle}_{a(t)} n.$$

note: ~~γto~~ $a(0) = 0$ &

$v = \gamma'(0) - \langle \gamma'(0), n \rangle n$ so $\gamma'(0) = v$

$\therefore$ $a'(0) = 0$ //

The curvature of γ is

$$\frac{1}{\|\gamma'(0)\|^2}\left\{\left\|\gamma''(0)-\gamma'(0)\frac{\langle\gamma'(0),\gamma''(0)\rangle}{\|\gamma''(0)\|^2}\right\|\right\}$$

(from before)

$$= \frac{|a''(0)|}{\|v\|^2} = \frac{|\langle\gamma''(0),n\rangle|}{\|v\|^2}$$

Now we show $\alpha(v,v) = \langle\gamma''(0),n\rangle$

Choose parameters ψ and write

$$\gamma(t) = \psi(x^1(t),x^2(t))$$

$$v = \gamma'(0) = \sum_i \frac{\partial\psi}{\partial x^i}(0)\frac{dx^i}{dt}(0)$$

$$\gamma''(0) = \sum_{i,j}\frac{\partial^2\psi}{\partial x^j\partial x^i}(0)\frac{\partial x^j}{\partial t}(0)\frac{\partial x^i}{\partial t}(0) + \sum\frac{\partial\psi}{\partial x^i}(0)\frac{d^2x^i}{dt^2}(0)$$

$$\therefore \langle\gamma''(0),n\rangle = \sum\left\langle\frac{\partial^2\psi}{\partial x^j\partial x^i},n\right\rangle v^jv^i + 0$$

$\left(\frac{\partial\psi}{\partial x^i}\perp n\right)$

$$= \alpha(v,v)$$

$$\therefore \frac{|\alpha(v,v)|}{\|v\|^2} = \text{curvature of } s+\operatorname{span}\{n,v\}\cap\Sigma$$

at s

So we have

Prop^n 5.3

If $v \in T_s\Sigma$ then

$$|\alpha(v,v)| = \|v\|^2 \times \text{curvature of } s + \text{span}\{n, v\} \cap \Sigma \text{ at } s.$$

The first fundamental form is the ~~[illegible]~~ bilinear form

$$T_s\Sigma \times T_s\Sigma \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto \langle v, w\rangle$$

the restriction of the inner product on $\mathbb{R}^3$ to $T_s\Sigma$.

Examples

The sphere of radius R.

$$\psi(\theta,\phi) = (R\cos\theta\sin\phi,\ R\sin\theta\sin\phi,\ R\cos\phi)$$

$$\frac{\partial\psi}{\partial\theta} = (-R\sin\theta\sin\phi,\ R\cos\theta\sin\phi,\ 0)$$

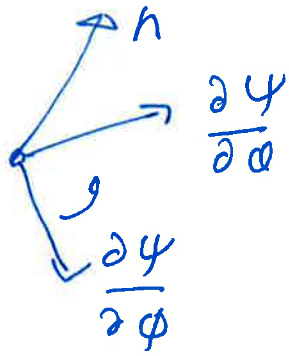
$$\frac{\partial\psi}{\partial\phi} = (R\cos\theta\cos\phi,\ R\sin\theta\cos\phi,\ -R\sin\phi)$$

$$\frac{\partial \psi}{\partial \phi} \times \frac{\partial \psi}{\partial \theta} = \begin{bmatrix} i & j & k \\ R\cos\theta\cos\phi & R\sin\theta\cos\phi & -R\sin\phi \\ -R\sin\theta\sin\phi & R\cos\theta\sin\phi & 0 \end{bmatrix}$$

$$= \left[R^2 \sin^2\phi \cos\theta \ , \ +R^2 \sin\theta \sin^2\phi \ , \ R^2 \sin\phi \cos\phi \right]$$

$$\therefore \ n = (\cos\theta\sin\phi \ , \ \sin\theta\sin\phi, \ \cancel{\sin\theta} \cos\phi)$$

outward normal



$$\frac{\partial^2 \psi}{\partial \theta^2} = (-R\cos\theta\sin\phi, -R\sin\theta\sin\phi \ , \ 0)$$

$$\frac{\partial^2 \psi}{\partial \phi \partial \theta} = (-R\sin\theta\cos\phi, \ R\cos\theta\cos\phi \ , \ 0)$$

$$\frac{\partial^2 \psi}{\partial \phi^2} = (-R\cos\theta\sin\phi \ , -R\sin\theta\sin\phi, \ -R\cos\phi)$$

$$\alpha\left(\frac{\partial\psi}{\partial\theta}, \frac{\partial\psi}{\partial\theta}\right) = \left\langle \frac{\partial^2\psi}{\partial\theta^2}, n\right\rangle$$

$$= -R\sin^2\phi$$

$$\alpha\left(\frac{\partial\psi}{\partial\theta}, \frac{\partial\psi}{\partial\phi}\right) = 0$$

$$\alpha\left(\frac{\partial\psi}{\partial\phi}, \frac{\partial\psi}{\partial\phi}\right) = -R$$

Note

$$n'\left(\frac{\partial\psi}{\partial\theta}\right) = \frac{\partial n'}{\partial\theta} = \frac{1}{R}\frac{\partial\psi}{\partial\theta}$$

$$\left(n'\left(\frac{\partial\psi}{\partial\theta}\right), \frac{\partial\psi}{\partial\theta}\right) = R\sin^2\phi$$

$$= -R\,\alpha\left(\frac{\partial\psi}{\partial\theta}, \frac{\partial\psi}{\partial\theta}\right)$$

Checking:

$$\alpha(v,w) = -(n'(v), w)$$

↓ End Lec 21

Example $g : \mathbb{R}^2 \to \mathbb{R}$

$$\Sigma = \{(x,y,z) \mid z = g(x,y)\}$$

graph of g

$$\psi(x,y) = (x, y, g(x,y))$$

$$\frac{\partial\psi}{\partial x} = \left(1, 0, \frac{\partial g}{\partial x}\right)$$

$$\frac{\partial\psi}{\partial y} = \left(0, 1, \frac{\partial g}{\partial y}\right)$$

$$\frac{\partial\psi}{\partial x} \times \frac{\partial\psi}{\partial y} = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)$$