Lecture 20

no. 1

Recall $\tilde{\gamma} : (a,b) \to C$

an arc-length parametrized oriented curve

Have 3 funct

$$T, N, B : C \to \mathbb{R}^3$$

defined as follow. If $c = \tilde{\gamma}(t)$

$$T(c) = \tilde{\gamma}'(t)$$

$$N(c) = \frac{T'(c)}{\|T'(c)\|} \qquad (T'(c) \neq 0)$$

$$B(c) = T(c) \times N(c)$$

Propⁿ 4.10 Let $\gamma : (a,b) \to C$ be a parametrized ~~curve~~ oriented curve & $c = \gamma(t)$ then

$$T(c) = \frac{\gamma'(t)}{\|\gamma'(t)\|} \qquad N(c) = \frac{T'(c)}{\|T'(c)\|}$$

Here $T'(c)$ means $T'(c) = \frac{d}{dt} T(\gamma(t))$

Proof Let $'$ be derivative wrt arc-length & $\tilde{\gamma}(t) = \gamma(p(t))$ be arc-length

$$\dot{\tilde{\gamma}}(t) = \gamma'(p(t)) \frac{dp}{dt}(t)$$

Proof

We have $\tilde{\gamma}(t) = \gamma(\rho(t))$

$$\therefore \dot{\tilde{\gamma}}(t) = \gamma'(\rho(t)) \frac{d\rho}{dt}(t)$$

As $\|\dot{\tilde{\gamma}}(t)\| = 1$ get $\dfrac{1}{\|\gamma'(\rho(t))\|} = \dfrac{d\rho}{dt}(t)$

$$\therefore \dot{\tilde{\gamma}}(t) = \frac{\gamma'(\rho(t))}{\|\gamma'(\rho(t))\|}$$

$\therefore$ ~~$T(t)$~~ If $c = \tilde{\gamma}(t) = \gamma(\rho(t)) = \gamma(s)$.

$$T(c) = \dot{\tilde{\gamma}}(t) = \frac{\gamma'(\rho(t))}{\|\gamma'(\rho(t))\|} = \frac{\gamma'(s)}{\|\gamma'(s)\|}$$

T is unit tangent vector in direction of orientation

Now we think of T as a function of t by $T(\tilde{\gamma}(t))$

or $\dot{T} = \dfrac{d}{dt} T(\tilde{\gamma}(t))$

or $T = T(\gamma(s))$

$$T'(s) = \frac{d}{ds} T(\gamma(s)) =$$

$\therefore \dot{T}(\tilde{\gamma}(t))$

$= T'(\gamma(\rho(t))) \frac{d\rho}{dt}(t)$

$\therefore \dfrac{\dot{T}}{\|\dot{T}\|}(\tilde{\gamma}(t))$

$= \dfrac{T'(\gamma(\rho(t)))}{\|T'(\quad)\|}$

This makes the helix calculation easier:

$$\gamma(t) = (a\sin(t),\ a\cos(t),\ bt)$$

$$\gamma'(t) = (a\cos(t),\ -a\sin t\ \ b)$$

$$\|\gamma'(t)\| = \sqrt{a^2+b^2}$$

$$T = \frac{\gamma'}{\|\gamma'\|} = \frac{1}{\sqrt{a^2+b^2}}\left(a\cos(t),\ -a\sin(t),\ b\right)$$

$$T' = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin(t)\ ;\ -a\cos t\ \ 0\right)$$

$$N = \frac{T'}{\|T'\|} = \left(-\sin(t),\ -\cos t,\ 0\right)$$

$$B = T\times N = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin t,\ -b\cos t,\ a\right)$$

all vectors at $c = (a\sin(t),\ a\cos(t),\ bt)$

$\therefore$ So at $c = \tilde{\gamma}(t) = \gamma(s)$ $(s = \rho(t))$

$$N(c) = \frac{\dot{T}}{\|\dot{T}\|}(\tilde{\gamma}(t)) = \frac{T'(\gamma(\cancel{\rho(t)}^{s}))}{\|T'(\gamma(\cancel{\rho(t)}_{s}))\|} = N(c).$$

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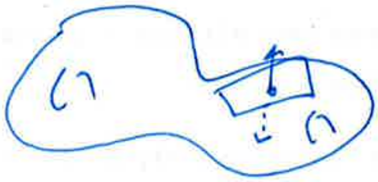
§5 Geometry of Surfaces (~~...~~ outline needs fixing.)

We have defined a surface to be a smooth 2-dim^{l} submanifold. We will assume from now on that it is in $\mathbb{R}^3$.

If $\Sigma \subseteq \mathbb{R}^3$ is a surface $T_x\Sigma$ is 2-dim^{l}. So $(T_x\Sigma)^\perp$ is 1 dim^{l}.

$\therefore (T_x\Sigma)^\perp - \{0\}$ has two connected component. i.e. there are two <u>normal</u> directions at x or two possible unit normal vectors. A continuous choice for all $x \in \Sigma$ of a normal direction is called an orientation of Σ or we say Σ is an oriented surface.



If $\varphi : U \to \Sigma$ is a parametrisation then $\frac{\partial\varphi}{\partial x^1} \times \frac{\partial\varphi}{\partial x^2}$ is normal and we say φ is an <u>oriented</u> parametrisation if this is in the chosen normal direction.

Example

Let $F : \mathbb{R}^3 \to \mathbb{R}$ & $\Sigma = F^{-1}(0)$.

$F'(x) \neq 0$ $\forall$ $x \in \Sigma$.

If $\operatorname{grad} F(x) = (F^{1\prime}(x), F^{2\prime}(x), F^{3\prime}(x))$

then $F'(x)(v) = \langle \operatorname{grad} F(x), v \rangle$.

so $\operatorname{grad} F(x) \perp T_x \Sigma$

& we can use it to define the normal direction.

If $n(x)$ is the unit normal to the oriented surface Σ then n is smooth because locally

$$n(\varphi(x_1, x_2)) = \frac{\frac{\partial \varphi}{\partial x^1}(x^1, x^2) \times \frac{\partial \varphi}{\partial x^2}(x^1, x^2)}{\left\| \frac{\partial \varphi}{\partial x^1}(x^1, x^2) \times \frac{\partial \varphi}{\partial x^2}(x^1, x^2) \right\|}$$

$\therefore n \circ \varphi(x^1, x^2) =$ which is smooth

20.[illegible]

The unit normal therefore defines a smooth function

$$n : \Sigma \longrightarrow S^2 \subseteq \mathbb{R}^3$$

called the <u>Gauss map</u> of Σ

<u>The second fundamental form</u>

Let $\Sigma \subseteq \mathbb{R}^3$ an oriented surface, $s \in \Sigma$ and n the oriented unit normal at s.

Let $\varphi : \underset{\text{open}}{U \subseteq \mathbb{R}^2} \longrightarrow \Sigma$ be a parametrisation with $\varphi(0,0) = s$.

Let $v, w \in T_s\Sigma$ so $v = \sum_{i=1}^{2} v^i \frac{\partial \varphi}{\partial x^i}(0,0)$.

$$w = \sum_{i=1}^{2} w^i \frac{\partial \varphi}{\partial x^i}(0,0)$$

We define the second fundamental form

$$\alpha : T_s\Sigma \times T_s\Sigma \longrightarrow \mathbb{R}$$

by $$\alpha(v,w) = \sum_{i,j} \left\langle \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(0,0), n \right\rangle v^i w^j$$

$\therefore$ $\cancel{x}\, n'(s) \in T_s \Sigma$

Consider

$\therefore$ $\langle n'(s)(v), w \rangle$

Think of n as $n \circ \psi$ with respect to parameter. ~~Then~~

$\cancel{\dfrac{\partial n \circ \psi}{\partial x^i}}\ \partial v^i$

Want ~~...~~ to calculate

$n'(s)(v)$

Choose α so that $\psi'(0,0)(\alpha) = v$

then $n'(s)(v) = n'(s)\,\psi'(0,0)(\alpha) = \sum \dfrac{\partial n \circ \psi}{\partial x^i}(0,0)\alpha^i$

But $\psi'(0,0)(e^i) = \dfrac{\partial \psi^i}{\partial x^i}(0,0)$

$\therefore$ $\psi'(0,0)(\alpha) = \sum_i \dfrac{\partial \psi^i}{\partial x^i}(0,0)\alpha^i$

$\therefore$ $n'(s)(v) = \dfrac{\partial n' \circ \psi}{\partial x^i}(0,0)\, v^i$

But if $v = \sum v^i \frac{\partial \psi}{\partial x^i}$

$$= \sum v^i \frac{\partial x}{\partial x^\ell} \frac{\partial \rho^\ell}{\partial x^i}$$

$$= \sum \left(\sum v^i \frac{\partial \rho^\ell}{\partial x^i} \right) \frac{\partial x}{\partial x^\ell}$$

This is coefficient of σ in $\frac{\partial \chi}{\partial x^\ell}$

basis

$\therefore$ α defined using χ basis is

$$\sum \left\langle \frac{\partial^2 \chi^i}{\partial x^k \partial x^\ell}, n \right\rangle \left(\sum v^i \frac{\partial \rho^\ell}{\partial x^i} \right) \left(\sum w^i \frac{\partial \rho^k}{\partial x^i} \right)$$

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(12) alternative

Recall ~~Note~~ that $n : \Sigma \to \mathbb{R}^3$ is smooth

~~as locally it looks like~~

$$\cancel{\pm \frac{\frac{\partial \psi}{\partial x^1} \times \frac{\partial \psi}{\partial x^2}}{\left\| \frac{\partial \psi}{\partial x^1} \times \frac{\partial \psi}{\partial x^2} \right\|}}$$

So we can ~~[illegible]~~ differentiate it

$$n'(s) : T_s \Sigma \to \mathbb{R}^3$$

as $\langle n, n \rangle = 1$ we have $\langle n'(s), n(s) \rangle = 0$

$$\therefore \langle n'(v), w\rangle = \sum_{i,j} \left\langle \frac{\partial n\circ\psi}{\partial x^i} v^i, \frac{\partial \psi}{\partial x^j} w^j \right\rangle$$

$$= \sum_{i,j} \left\langle \frac{\partial n\circ\psi}{\partial x^i}, \frac{\partial \psi}{\partial x^j} \right\rangle v^i w^j$$

But $\left\langle n\circ\psi, \frac{\partial \psi}{\partial x^j} \right\rangle = 0$

$$\therefore \left\langle \frac{\partial n\circ\psi}{\partial x^i}, \frac{\partial \psi}{\partial x^j} \right\rangle + \left\langle n, \frac{\partial^2 \psi}{\partial x^j \partial x^i} \right\rangle = 0$$

$$\therefore \langle n'(v), w\rangle = -\sum_{i,j} \left\langle \cancel{\frac{\partial n\circ\psi}{\partial x^i}} \frac{\partial^2 \psi}{\partial x^j \partial x^i}, n \right\rangle v^i v^j$$

$$= -\alpha(v,w)$$

~~So~~ LHS doesn't depend on parameta!

~~**Prop^n. 5.1** Let $\alpha : T_s\Sigma \times T_s\Sigma \to \mathbb{R}$ be the second fundamental form on Σ.~~

~~Then~~

~~① α is bilinear & symmetric~~

~~ie $\alpha(v,w) = \alpha(w,v)$~~

~~$\alpha(\lambda v + \mu u, w) = \lambda\alpha(v,w) + \mu\alpha(v,w)$~~

~~$\alpha(v, \lambda u + \mu w) = \lambda\alpha(v,u) + \mu\alpha(v,w)$~~