Geometry of Surface

Introduction

A surface is a special kind of (smooth) subset of $\mathbb{R}^3$.

Example:

$S^2 = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}$

or

not smooth:

A surface is an example of a manifold and in particular of a submanifold of $\mathbb{R}^3$. ($\mathbb{R}^3$ is also a manifold.)
We want to prove the Gauss-Bonnet Theorem.

It is possible to state this without defining too much so let we do that.

1. Start with curves in \( \mathbb{R}^2 \)

\[ \gamma(t), \quad \gamma'(t), \quad \gamma''(t), \quad \gamma'''(t) \]

Define curvature as \( \| \gamma''(t) \| \).

Also radius of circle of "best fit".

2. Now consider \( S \) a surface
\( p \in \Sigma \)  
\( \hat{n}_p = \text{unit normal} \)

lead to choice a directa for \( \hat{n}_p \) (called an orientation)

\[ T_p \Sigma = \{ v \in \mathbb{R}^3 : \langle v, \hat{n}_p \rangle = 0 \} = (\hat{n}_p)^\perp \]

\[ \text{tangent space to } \Sigma \text{ at } p \]

If \( v \in T_p \Sigma \) \( \{ v, \hat{n}_p \} \) span a 2-dim space, and \( \gamma \subset \Sigma \) is a curve, say.

Define \( \tau (v, v) = \text{curvature of } \gamma \)

There is a 

**Remarkable fact**

form \( \tau \) s.t. \( \tau (v, v) = \text{result ason} \)

let \( \lambda_1 = \text{largest curvature} \) \( \lambda_2 = \text{smallest curvature} \)

\[ K = \frac{\lambda_1 \lambda_2}{4} = \text{Gaussian curvature} \]
\[ H = \frac{1}{2} (\lambda_1 + \lambda_2) = \text{mean curvature} \]

Gauss – Bonnet:

\[ \frac{1}{2\pi} \int \int K \, ds = 2 - 2g \]

\# of holes
\( g = 0 \)
\( g = 1 \)
\( g = 2 \)

**NB** LHS - differential change if we deform surface

RTS - topological doesn't change if we deform surface.

Example of an index theorem - fundamental

Classifying a thm for compact, oriented surface

\[ \text{Determined by genus - } 2g - 2 = \text{Euler class} \]

Mean curvature \( H = 0 \) - minimal surface locally has smallest area. "soap bubble"

What is \( \Pi \) ?

**Triangle:** \( x, y, z \)

\[ f(0,0) = 0 \]

\[ f_x(0,0) = 0 \]

\[ f_y(0,0) = 0 \]

\[ f_{xx}(0,0) = 0 \]

\[ f_{yy}(0,0) = 0 \]

\[ f_{xy}(0,0) = 0 \]

\[ f_{xyz}(0,0) = 0 \]

Locally \( \delta \Sigma : z = f(x, y) \)
\[ H = \left( \frac{\partial^2 f}{\partial x \partial y} \right) \text{ "Hessian" } \]

We need to go back and develop the definition of submanifolds & tools for solving these problems.

Note: Familiarity with these concepts is also important.

Other kinds of geometry

One way to define a submanifold is to say it is defined by an eq
\[ f(x, y, z) = 0 \] where \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \)

is differentiable & \( f'(x, y, z) \neq 0 \) if
\[ E.g. \quad f(x, y, z) = x^2 + y^2 + z^2 - 1. \]

Other geometries arise by taking different kinds of functions.

Eg \( f(x, y, z) \) a polynomial.

Then the properties of \( \mathbb{R}^3 \) are important. (More generally \( F[x_1, \ldots, x_n] \) a field \( F \) a field in algebraic geometry (Fields & Geometry) \( F^n \).
§1. Review

Notation
- \( \mathbb{R} \) real numbers
- \( \mathbb{N} = \{1, 2, 3, \ldots \} \) natural numbers
- \( \mathbb{R}^n = \text{n-tuples } x = (x', \ldots, x^n) \) (vectors)

No bold face, underline, arrows etc for vectors

If \( x = (x', \ldots, x^n) \), \( y = (y', \ldots, y^n) \)

\[ \langle x, y \rangle = x'y' + \ldots + x^n y^n \] inner product

\[ \| x \| = \sqrt{\langle x, x \rangle} = \sqrt{(x')^2 + \ldots + (x^n)^2} \] norm or length of \( x \).

NB \( \| x \| = 0 \iff x = 0 \)

Those satisfy:

\[ \langle x, y \rangle \leq \| x \| \| y \| \] Cauchy's

Exercise: \( \exists \gamma \langle x - \langle x', y' \rangle y', x - \langle x', y' \rangle y' \rangle \geq 0 \) inequality
(Square and Cauchy) \( \| x + y \| \leq \| x \| + \| y \| \) \( \Delta \) inequality

\[ \text{max} \{ \| x' \|, \ldots, \| x^n \| \} \leq \| x \| \leq \sqrt{n} \text{ max} \{ \| x' \|, \ldots, \| x^n \| \} \]

If \( x \in \mathbb{R}^n, \varepsilon > 0 \)

\[ B(x, \varepsilon) = \{ y \in \mathbb{R}^n \mid \| x - y \| < \varepsilon \} \]

\( \frac{1}{\| x \|} = \| x \| \)

\[ \| x \| \leq \| x \| \leq \sqrt{n} \text{ max} \{ \| x' \|, \ldots, \| x^n \| \} \]

Exercise

\[ B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon) \]
A subset $U \subseteq \mathbb{R}^n$ is called open if

$$\forall x \in U \exists \varepsilon > 0 \text{ s.t. } B(x, \varepsilon) \subseteq U.$$  

Note: $\varepsilon$ depends on $x$ usually.

A sequence in $\mathbb{R}^n$ is a function $n = \{1, 2, \ldots, y \to \mathbb{R}^n$ usually we write $x = (x_n)_{n \in \mathbb{N}}$ or just $x_n$.

A sequence $\{x_n\}$ has limit $x$ if $\forall \varepsilon > 0$ \exists $N$ s.t. $\forall n > N$ $\|x_n - x\| \leq \varepsilon$.

Or if $\lim_{n \to \infty} \|x_n - x\| = 0$.

In such a case $x$ is unique & we write $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

Lemma: If $x_n \to x$ then $\lim_{m,n \to \infty} \|x_m - x_n\| = 0$.

Proof:

$$0 \leq \|x_m - x_n\| = \|x_m - x + x - x_n\| \leq \|x_m - x\| + \|x_n - x\|.$$