Geometry of Surfaces (p1) Lec 1 26/7/2011

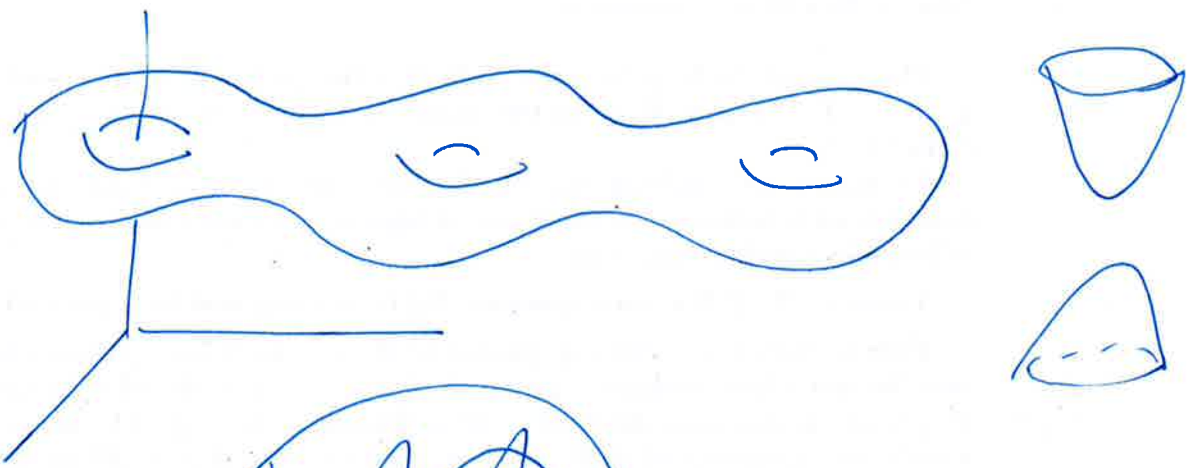
§0 Introduction

* Handouts
* Consulting time next lectures
*

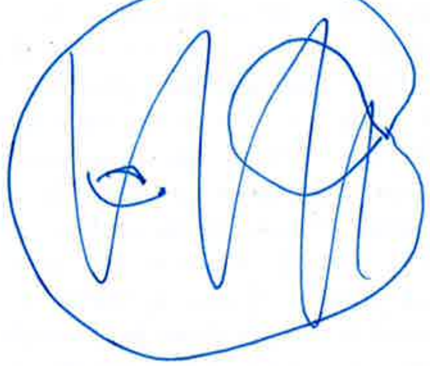
A surface is a special kind of (smooth) subset of $\mathbb{R}^3$.

EG: $S^2 = \{(x, y, z) \mid x^2+y^2+z^2=1\}$

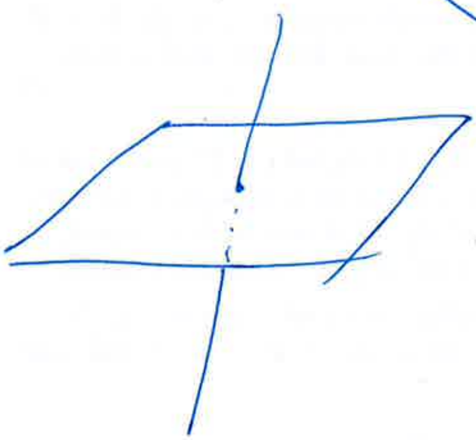
or

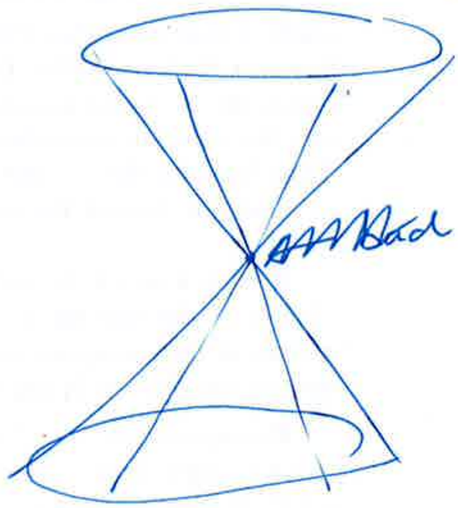


not smooth:



"2-dimensional"



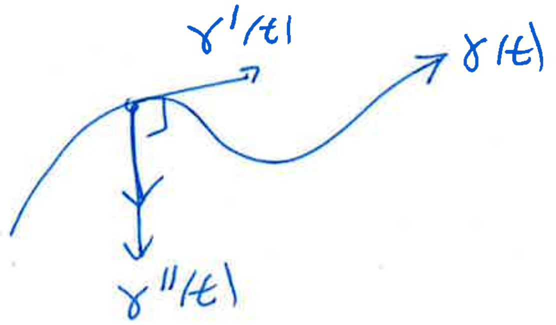


A surface is an example of a manifold and in particular of a submanifold of $\mathbb{R}^3$. ($\mathbb{R}^3$ is also a manifold.)

We want to prove the <u>Gauss-Bonnet</u>
<u>Theorem</u>

It is possible to state this without defining too much so let us do that.

① Start with <u>curves</u> in $\mathbb{R}^2$

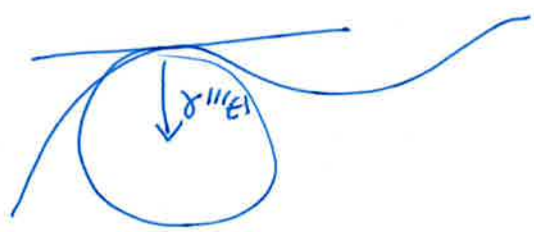


$\gamma : [a,b] \to \mathbb{R}^2$
differentiable

Assume $\langle \gamma'(t), \gamma'(t) \rangle = 1$ $\quad \therefore 2\langle \gamma''(t), \gamma'(t) \rangle = 0$

$\gamma'(t)$ velocity
$\gamma''(t)$ acceleration

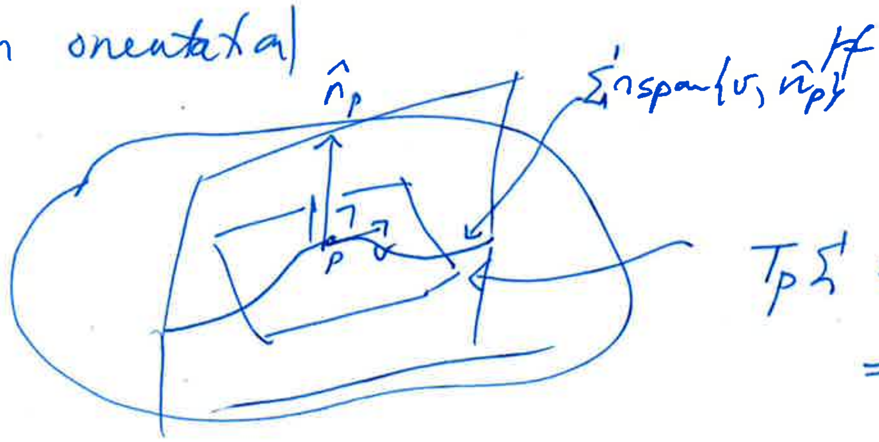
Define <u>curvature</u> as $\|\gamma''(t)\|$.
Also radius of circle of "best fit"



② Now consider Σ a surface

$p \in \Sigma$ $\hat{n}_p$ = unit normal

need to choose a direction for $\hat{n}_p$ / called an orientation



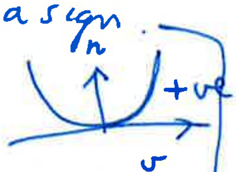
$T_p\Sigma = \{v \in \mathbb{R}^3 / \langle v, \hat{n}_p \rangle = 0\}$
$= (\hat{n}_p)^{\perp}$

tangent space to Σ at p

imagine you are $\hat{n}_p$

If $v \in T_p\Sigma$ $\{v, \hat{n}_p\}$ span a 2-dim space and $\cap$ Σ in a curve γ_v say.

Define $\pi(v,v)$ = curvature of γ_v ← can make it have a sign.



Remarkable fact ~~π is~~ There is an ~~bilinear symmetric~~ quadratic, symmetric form π s.t. $\pi(v,v)$ = result above

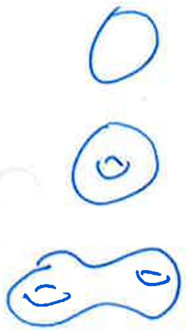
Let λ_1 = largest curvature λ_2 = smallest curvature — principal curvatures

$K = \lambda_1\lambda_2$ = Gaussian curvature

$H = \frac{1}{2}(\lambda_1 + \lambda_2)$ = mean curvature

Gauss-Bonnet : $\frac{1}{2\pi}\iint K\, dS = 2 - 2g$

↑ # of holes



$g=0$

$g=1$

$g=2$

$\boxed{\text{NB}}$ LHS – differential change if we ~~squish~~ deform surface

RHS – topological doesn't change if we deform surface.

Example of an index thm – ~~fundamental~~

Classification thm for compact ~~closed~~, orientable surface

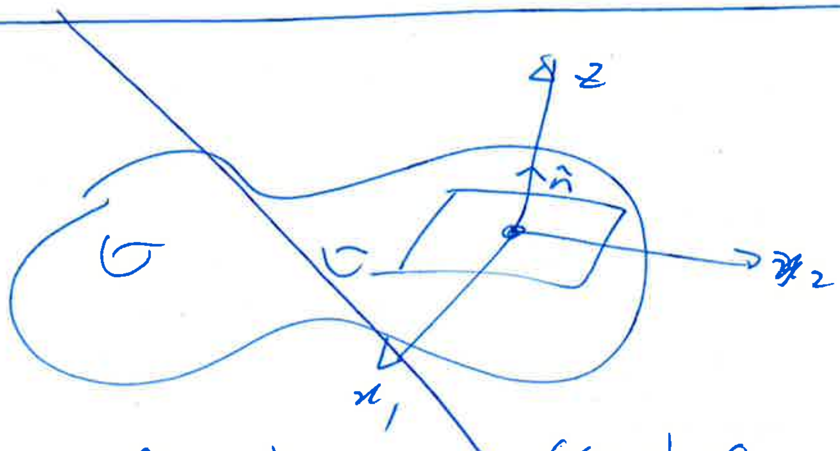
up to homeo^M --

Determined by genus. – $2g-2$ = Euler class.

TRIANGULATION χ-class
#faces − #edges + #verts

Mean curvature $H=0$ – minimal surface. locally has smallest area. "soap bubbles".

What is π?



locally $\Sigma : z = f(x_1, x_2)$ ~~+~~

$f(0,0)=0$

$f_x(0,0)=0$

$f_y(0,0)=0$

~~$z = \frac{\partial f}{\partial x \partial y} xy + \frac{1}{2}\frac{\partial f}{\partial x^2} x^2 + \frac{1}{2}\frac{\partial f}{\partial y} y^2 + \ldots$~~

$$H = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}(0,0) \right) \quad \text{"Hessian"}$$

We need to go back and develop the definition of submanifolds & tools for solving these problems.

~~Note formally if we change the definition of~~

Other kinds of geometry

One way to define a submanifold is to say it is defined by an eqⁿ $f(x,y,z) = 0$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable & $f'(x,y,z) \neq 0$ if $f(x,y,z)=0$

E.G $f(x,y,z) = x^2+y^2+z^2-1$.

Other geometries arise by taking different kinds of functions.

Eg $f(x,y,z)$ a polynomial.

Then the properties of $\mathbb{R}[x,y,z]$ are important. (More generally $\mathbb{F}[x_1, \dots, x_n]$ a ~~field~~ $\mathbb{F}$ a field $\to$ algebraic geometry (Fields & Geometry) $\mathbb{F}^n$

§1. Review

Notation $\mathbb{R}$ real numbers

$\mathbb{N} = \{1, 2, 3, \dots\}$ natural numbers

$\mathbb{R}^n =$ n-tuples $x = (x^1, \dots, x^n)$ (vectors)

no boldface, underline, arrows etc for vectors

If $x = (x^1, \dots, x^n)$ $y = (y^1, \dots, y^n)$

$\langle x, y \rangle = x^1 y^1 + \dots + x^n y^n$ inner product

$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{(x^1)^2 + \dots + (x^n)^2}$ norm or length of x.

NB $\|x\| = 0 \iff x = 0$

These satisfy:

$\langle x, y \rangle \leq \|x\| \, \|y\|$ Cauchy's inequality

Exercise  $\left\langle x - \frac{\langle x, y \rangle y}{\langle y, y \rangle}, x - \frac{\langle x, y \rangle y}{\langle y, y \rangle} \right\rangle \geq 0$

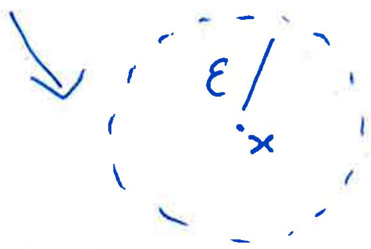
(square & use Cauchy) $\|x + y\| \leq \|x\| + \|y\|$ Δ inequality

$\max\{|x^1|, \dots, |x^n|\} \leq \|x\| \leq \sqrt{n} \max\{|x^1|, \dots, |x^n|\}$

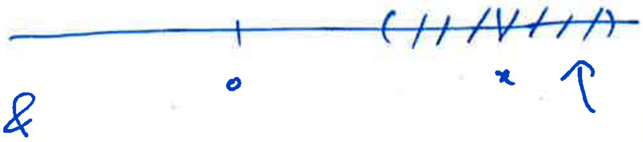
Exercise

If $x \in \mathbb{R}^n$, $\varepsilon > 0$

$B(x, \varepsilon) = \{ y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon \}$



If $n = 1$ $\|x\| = |x|$



& $B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$

A subset $U \subseteq \mathbb{R}^n$ is called <u>open</u> if $\forall x \in U \ \exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq U$.

$\boxed{\text{NB } \varepsilon \text{ depends on } x \text{ usually.}}$

A <u>sequence</u> in $\mathbb{R}^n$ is a function

$$\mathbb{N} = \{1, 2, \dots\} \longrightarrow \mathbb{R}^n$$

usually we write $\{x_n\}_{n=1}^{\infty}$ or ~~just~~ ~~$\{x_1, x_2, \dots\}$ or~~ just x_n.

DECIMAL PLACES

A sequence $\{x_n\}$ has limit x if $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n \geq N$ $\|x_n - x\| \leq \varepsilon$

or if $\lim_{n \to \infty} \|x_n - x\| = 0$. $\leftarrow$ SEQUENCE OF REAL NUMBERS

In such a case x is <u>unique</u> & we write $\lim_{n \to \infty} x_n = x$ or $x_n \longrightarrow x$

End Lecture 1 2011

~~<u>Lemma</u> If $x_n \to x$ then $\lim_{m,n \to \infty} \|x_m - x_n\| = 0$~~

~~<u>Proof</u>~~

~~$0 \leq \|x_m - x_n\| = \|(x_m - x) - (x_n - x)\|$~~

~~$\leq \|x_m - x\| + \|x_n - x\|$~~