$$\tilde\gamma(t) = \gamma(p(t))$$
↑ arc-length parametrised.

Hence

$$\tilde\gamma''(t) = \frac{\gamma''(p(t))}{\|\gamma'(p(t))\|^2} - \frac{\gamma'(p(t))\,\langle\gamma''(p(t)), \gamma'(p(t))\rangle}{\|\gamma'(p(t))\|^4}$$

So at $\tilde\gamma(t) = \gamma(p(t))$ we have $K(\tilde\gamma(t)) = \|\tilde\gamma''(t)\|$

$\therefore$ at $\tilde\gamma(t) = \gamma(p(t))$

$$\therefore\ K = \frac{1}{\|\gamma'\|^2}\left(\left\| \gamma'' - \frac{\gamma'\langle\gamma'', \gamma'\rangle}{\|\gamma'\|^2} \right\| \right)$$

$$= \frac{1}{\|\gamma'\|^2}\left(\|\gamma''\|^2 - 2\frac{\langle\gamma'', \gamma'\rangle^2}{\|\gamma'\|^2} + \frac{\|\gamma'\|^2\langle\gamma'', \gamma'\rangle^2}{\|\gamma'\|^4} \right)^{\frac{1}{2}}$$

evaluate at $s = p(t)$

$$K(\gamma(s)) = \boxed{\frac{1}{\|\gamma'\|^2}\left(\|\gamma''\|^2 - \frac{\langle\gamma'', \gamma'\rangle^2}{\|\gamma'\|^2} \right)^{\frac{1}{2}}} \qquad \boxed{s = p(t)}$$

Prop^n 4.7 If $\gamma : (a,b) \to \mathbb{C}$ is a parametrised curve at the point $c = \gamma(s)$ we have

$$K(c) = \frac{1}{\|\gamma'(s)\|^2}\left(\|\gamma''(s)\|^2 - \frac{\langle\gamma''(s), \gamma'(s)\rangle^2}{\|\gamma'(s)\|^2} \right)^{\frac{1}{2}}$$

Example   Helix   19.2.~~18~~

$$\gamma(t) = (a\cos t, a\sin t, bt) \qquad a>0$$

$$\gamma'(t) = (-a\sin t, a\cos t, b)$$

$$\gamma''(t) = (-a\cos t, -b\sin t, 0).$$

$$\|\gamma'(t)\| = \sqrt{a^2+b^2}$$

$$\|\gamma''(t)\| = \cancel{\sqrt{a^2+b^2}} \; a.$$

$$\langle \gamma''(t), \gamma'(t)\rangle = 0$$

$$\therefore \quad \kappa = \frac{1}{(a^2+b^2)}\left(a \right) = \frac{a}{a^2+b^2}$$

~~Assume now our curve is in $\mathbb{R}^3$~~

Standard ~~notation~~ is to let ~~T be~~

$T = \gamma'(t)$ where γ is an arc-length

(unit ~~normal vec~~ tangent vector)

parametrisation. The $\langle T, T\rangle = 1$

$$\left\langle T, \frac{dT}{dt}\right\rangle = 0. \qquad \text{If } T' \neq 0$$

∴ we let $N = \dfrac{T'}{\|T\|}$ and call it

the principal normal vector to the curve

Assume now $C \subseteq \mathbb{R}^3$ & $\gamma : (a,b) \to C$ arc-length parametrisation

Define $T = \gamma'(t)$ unit tangent vector

$\langle T, T \rangle = 1 \quad \therefore \quad \langle T, T' \rangle = 0$

If $T' \neq 0$ let

$N = \dfrac{T'}{\|T'\|}$ principal normal vector

& $B = T \times N$ unit binormal vector

Propn 4.8 If $\gamma : (a,b) \to C$ is an arc-length parametrised curve ~~with~~ ~~γ''~~ we define

$T = \gamma'$ unit tangent vector

$N = \dfrac{T'}{\|T'\|}$ (if $T' \neq 0$) principal normal vector

$B = \cancel{N \times T}\ T \times N$ unit binormal vector

Ex Use the triple product

$a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$

to show that $T = N \times B$

$N = B \times T.$

19.4.~~18~~19.

~~The unit binormal vector B is defined by~~ ~~Prop^n 4.8 summary of T, N, B~~

~~$B = T \times N$~~

NB: (T, N, B) are an orthonormal basis for $\mathbb{R}^3$ at each point of C.

Example Helix

$$T = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t,\ a\cos t,\ b).$$

$$T' = \frac{1}{\sqrt{a^2+b^2}}(-a\cos t,\ -a\sin t,\ 0)$$

$(-a\sin t,\ a\cos t,\ b)$

$$N = (-\cos t,\ -\sin t,\ 0)$$

$$B = T \times N = \frac{1}{\sqrt{a^2+b^2}}(b\sin t,\ -b\cos t,\ a).$$

Frenet formulae

Let us use $\dot{}$ to mean derivative wrt arc-length.

19.5 ~~18.10~~

Then $\dot{T} = \|\dot{T}\| N = \kappa N$ $\boxed{\dot{T} = \kappa N}$

We have $\langle N, N\rangle = 1$ $\therefore$ $\langle N, \dot{N}\rangle = 0$

$\therefore$ $\dot{N} = \sigma T + \tau B$

Also $\langle T, N\rangle = 0$ $\therefore$

$0 = \langle \dot{T}, N\rangle + \langle T, \dot{N}\rangle$ or

$\kappa = \kappa\langle N, N\rangle = -\langle T, \dot{N}\rangle$

$\therefore$ $\langle T, \dot{N}\rangle = -\kappa$

$\therefore$ $\sigma = -\kappa$

$\therefore$ $\boxed{\dot{N} = -\kappa T + \tau B}$

Now $B = T \times N$

$\therefore$ $\dot{B} = \dot{T} \times N + T \times \dot{N}$ $\qquad \dot{T} = \kappa N$ $\therefore$ $\dot{T} \times N = 0$

$= T \times \dot{N}$

B, T, N o/n

B = T × N

triple product

T = N × B

N = B × T

$= T \times (-\kappa T + \tau B).$

$= \tau\,(T \times B)$

$= -\tau N$

$\therefore$ $\boxed{\dot{B} = -\tau N}$

Call τ the torsion of the curve

If $\tau = 0$ $\dot{B} = 0$ then B is constant

and ~~mo~~ γ is in the plane spanned

by T, N. ie motion is planar

(If $\dot{T} = 0$ $N = 0$ $K = 0$ – motion is in a straight line)

Prop^n 4·8 | Frenet formula |

$$\dot{T} = K N$$

$$\dot{N} = -K T + \tau B$$

$$\dot{B} = -\tau N.$$

~~Note If T, N, B : $(a,b) \to \mathbb{R}^3$~~

~~satisfying the Frenet formula you~~ can

~~find a curve γ for which this~~ is

~~its T, N, B. See Appendix~~ to

~~Schaum Diff~~

~~If $K, \tau : (0, s) \to \mathbb{R}$.~~

~~Can show that if~~

Example

~~Helix~~ Torsion of helix

We have

~~$\tau = \frac{1}{\sqrt{a^2+b^2}}$~~

$$\boxed{\dot{B} = -\tau N}$$

$$B = \frac{1}{\sqrt{a^2+b^2}} (b \sin t, \; -b\cos t, \; a).$$

$$B' = \frac{b}{\sqrt{a^2+b^2}} (\cos t, \; \sin t, \; 0).$$

Assume ρ is arc-length parametrisation

$$\rho' = \frac{1}{\|\gamma'\|^2} = \frac{1}{\sqrt{a^2+b^2}}$$

$$\dot{B} = B' \frac{d\rho}{dt} = B' \frac{1}{\sqrt{a^2+b^2}}$$

$$= \frac{b}{a^2+b^2} (\cos t, \sin t, 0)$$

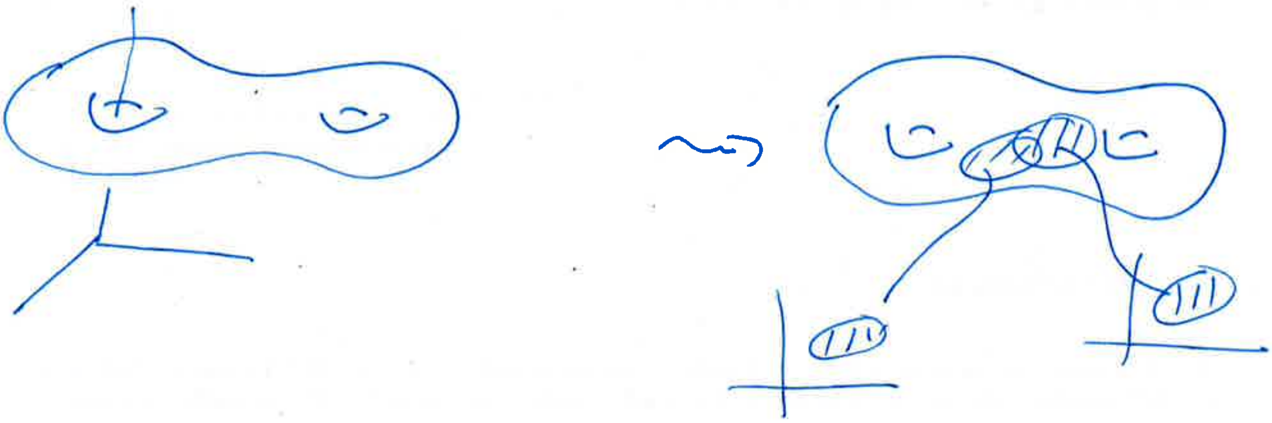
$$= -\left(\frac{b}{a^2+b^2}\right) N$$

$$\therefore \boxed{\tau = \frac{b}{a^2+b^2}}$$

Note $b=0$ get planar $(a\sin t, a\cos t, 0)$.

Aside :

Manifold



co-ordinate charts

inverse of parametrisation what about tangent space

what is a tangent vector?

a submanifold is a manifold

Same note from the level IV course

on my webpage under Publications — other

End Lec 19