Lecture 18

Mid-test only up to material before curves.

Correction Def^n 4.2

A parametrised curve C is a curve with a parametrisation $\gamma : (a,b) \to C$ with $\gamma(a,b) = C$.

So really a parametrised curve is a pair (C, γ).

Note We can say C is parametrisable if $\exists$ a ~~ϵ $\gamma : (a,b)$ with $\gamma($~~ parametrisation $\gamma : (a,b) \to C$ with $\gamma(a,b) = C$ but we don't need this notion. //

Recall we def'd $\gamma : (a,b) \to C$ to be a parametrisation by arc-length if it was a parametrisation & $\|\gamma'(t)\| = 1$ $\forall t \in (a,b)$.

Propⁿ 4.4 If $\gamma : (a,b) \to \mathbb{R}$ C is a parametrised curve then C has a parametrisation by arc-length

Proof Fix $t_0 \in (a,b)$

Consider $\sigma(t) = \int_{t_0}^{t} \|\gamma'(\xi)\|\, d\xi$

Then $\sigma'(t) = \|\gamma'(t)\| > 0$

& infinitely differentiable as $\gamma'(t) \neq 0$.

So $\sigma : (a,b) \to (c,d)$ is increasing, smooth so 1-1 & onto & by inverse function thᵐ the inverse is smooth.

Let $\tilde{\gamma} : (c,d) \to \mathbb{R}^n$

$$\tilde{\gamma}(s) = \gamma \circ \sigma^{-1}(s).$$

$$\tilde{\gamma}'(s) = \gamma'(\sigma^{-1}(s))\, \sigma^{-1\,\prime}(s)$$

$$\& \quad \sigma^{-1\,\prime}(s) = \frac{1}{\sigma'(\sigma^{-1}(s))} = \frac{1}{\|\gamma'(\sigma^{-1}(s))\|}$$

chainrule

$$\therefore \|\tilde{\gamma}'(s)\| = \|\gamma'(\sigma^{-1}(s))\| \, \|\sigma^{-1\,\prime}(s)\|$$

$$\| \quad \frac{1}{\|\gamma'(\sigma^{-1}(s))\|}$$

$$= 1$$

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Lemma 4.4

If γ, $\tilde{\gamma}$ are two parametrisations by arc-length with the same orientation then $\exists\, t_0$ s.t.

$$\gamma(t) = \tilde{\gamma}(t + t_0)$$

Proof Assume $\gamma : (a,b) \to C$

$\tilde{\gamma} : (c,d) \to C$. Then $\exists\, \rho : (a,b) \to (c,d)$ a diffeo^m s.t. $\gamma = \tilde{\gamma} \circ \rho$

$\therefore$ ~~$\frac{d\gamma}{dt} = \frac{d\tilde{\gamma}}{}$~~

$$\gamma'(t) = \tilde{\gamma}'(\rho(t)) \cdot \rho'(t)$$

But $\|\gamma'(t)\| = \|\tilde{\gamma}'(\rho(t))\| = 1$ $\therefore$ $\rho'(t) = \pm 1$

But same orientation $\Rightarrow +1$.

$\therefore$ $\rho(t) = t + t_0$.

Note $\gamma'(t) = \tilde{\gamma}'(t+t_0)$, $\gamma''(t) = \tilde{\gamma}''(t+t_0)$ etc.

Example $C = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x^2+y^2 = R \\ (x,y) \neq (R,0) \end{array} \right\}$

$\gamma : (0, 2\pi) \to C$

$\gamma(\theta) = (R\cos\theta, R\sin\theta)$

$$\gamma'(\theta) = (-R\sin\theta, R\cos\theta)$$

$$\|\gamma'(\theta)\| = R$$ not parametrised by arc-length.

$$\tilde{\gamma}(\theta) = (R\cos \theta/R, R\sin \theta/R)$$

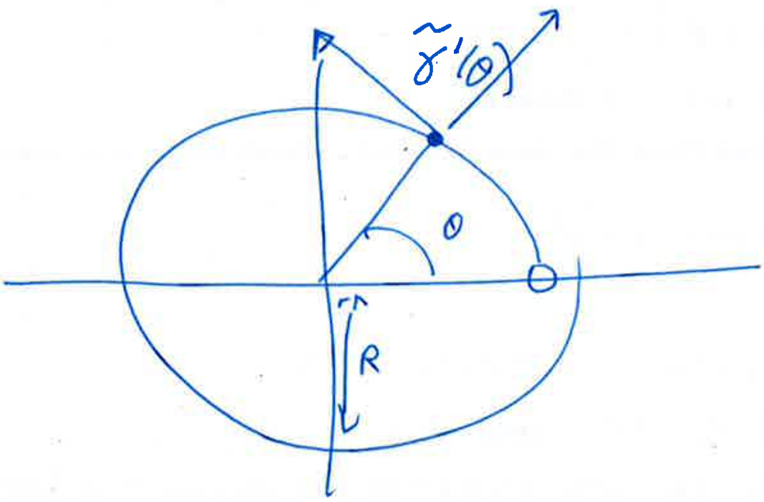
$$\tilde{\gamma} : (0, 2\pi R) \longrightarrow \cancel{R}\, C$$

$$\tilde{\gamma}(\theta) = (-\sin\theta, \cos\theta)$$

$$\|\tilde{\gamma}(\theta)\| = 1$$

parametrised by arc length

$$\int_0^{2\pi R} \|\tilde{\gamma}(\theta)\|^2 \, d\theta = 2\pi R$$ as we expect



We are going to interested in the derivative of $\tilde{\gamma}'(\theta)$

$$\tilde{\gamma}''(\theta) = \frac{1}{R}(\cos \theta/R, \sin \theta/R)$$

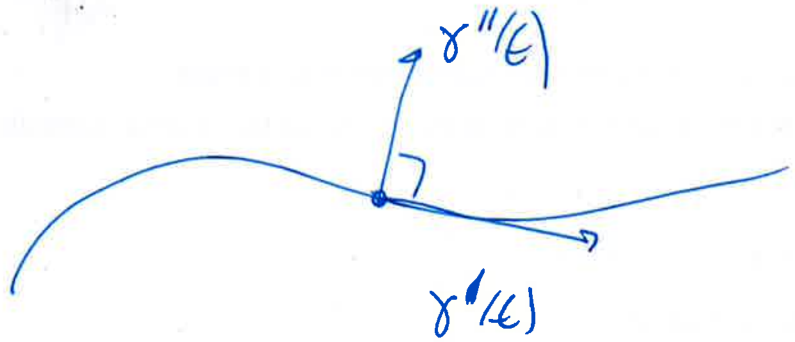
$$\|\tilde{\gamma}''(\theta)\| = \frac{1}{R}$$

Lemma 4.5/f γ ~~is a~~ $: (a,b) \to \mathbb{C}$ is ~~a parametrised curve~~ curve parametrised by arc-length the

$$\langle \gamma''(t), \gamma'(t) \rangle = 0$$

Proof $\frac{d}{dt}$ $\langle \gamma'(t), \gamma'(t) \rangle = 1$

$\therefore 2 \langle \gamma''(t), \gamma'(t) \rangle = 0$. //



Defn 4.6 Let $\gamma : (a,b) \to \mathbb{C}$ be a curve parametrised by arc-length. Define $\kappa : C \to [0, \infty)$ by

$$\kappa(\gamma(t)) = \|\gamma''(t)\|$$

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and call it the curvature of C.

Notice that κ is actually independent of the choice of arc-length parametrisation & hence depends only on C.

What if we know C as parametrised but not arc-length parametrised?

Assume $\gamma : (a,b) \to C$

& ~~$\rho : (a,b) \to (c,d)$~~ $\rho : (c,d) \to (a,b)$

& that $\tilde{\gamma}(t) = \gamma(\rho(t))$ is an arc-length parametrisation. Then

$$\tilde{\gamma}'(t) = \gamma'(\rho(t)) \cdot \rho'(t)$$

& $$\tilde{\gamma}''(t) = \gamma''(\rho(t))(\rho'(t))^2 + \gamma' \circ \rho(t) \rho''(t)$$

But $\|\tilde{\gamma}'(t)\| = 1$ $\therefore$

$$\rho'(t) = \|\gamma' \circ \rho\|^{-1} = \langle \gamma' \circ \rho, \gamma' \circ \rho \rangle^{-\frac{1}{2}}$$

$$\rho''(t) = -\tfrac{1}{2} \langle \gamma' \circ \rho, \gamma' \circ \rho \rangle^{-3/2} \, 2 \langle (\gamma'' \circ \rho \, \rho'), (\gamma' \circ \rho) \rangle$$

$$= - \frac{\langle \gamma'' \circ \rho, \gamma' \circ \rho \rangle}{\|\gamma' \circ \rho\|^4}$$

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Hence

$$\tilde{\gamma}''(t) = \frac{\gamma''(p(t))}{\|\gamma'(p(t))\|^2} - \frac{\gamma'(p(t))\,\langle \gamma''\circ p(t), \gamma'(p(t))\rangle}{\|\gamma'(p(t))\|^4}$$

So at $\tilde{\gamma}(t) = \gamma(p(t))$ we have

$$\kappa(\cdot = \frac{1}{\|\gamma'\|^2}\left(\left\| \gamma'' - \gamma' \frac{\langle \gamma'', \gamma' \rangle}{\|\gamma'\|^2} \right\| \right.$$

$$= \frac{1}{\|\gamma'\|^2}\left(\|\gamma''\|^2 - 2\frac{\langle \gamma'', \gamma'\rangle^2}{\|\gamma'\|^4} + \frac{\|\gamma'\|^2\langle \gamma'', \gamma'\rangle^2}{\|\gamma'\|^4} \right)^{\frac{1}{2}}$$

$$= \frac{1}{\|\gamma'\|^2}\left(\|\gamma''\|^2 - \frac{\langle \gamma'', \gamma'\rangle^2}{\cancel{4}\|\gamma'\|^4} \right)^{\frac{1}{2}}$$

~~**Prop^n 4.17** If $\gamma : (a,b) \to \mathbb{C}$ is a parametrised curve at the point $\gamma(s)$ we have~~

$$\cancel{\kappa(\gamma(s)) = \frac{1}{\|\gamma'(s)\|^2}\left(\|\gamma''(s)\|^2 - \frac{\langle \gamma''(s), \gamma'(s)\rangle}{\|\gamma'(s)\|^4} \right)^{\frac{1}{2}}}$$