Lecture 17

Let us back up a bit & redo this.

Prop^n 3.13. Let $g : \underset{V \text{ open}}{\mathcal{U}} \to \mathbb{R}^N$ ($\mathcal{U} \subseteq \mathbb{R}^n$) be a smooth function with $g(\mathcal{U}) \subseteq S$ a submanifold. If $f : S \to \mathbb{R}^M$ is smooth then

$f \circ g : \mathcal{U} \to \mathbb{R}^M$ is smooth

Proof:

We show that each $x \in \mathcal{U}$ has an open nbd ~~V~~ V such that $(f \circ g)|_V : V \to \mathbb{R}^M$ is smooth.

Consider $g(x) \in S$. As f is smooth $\exists\, W \subseteq \mathbb{R}^N$ open with $g(x) \in W$ & $\tilde{f} : W \to \mathbb{R}^M$ smooth such that $\tilde{f}|_{W \cap S} = f|_{W \cap S}$.

Let $V = g^{-1}(W)$ open. Then

$$(f \circ g)|_V = (\tilde{f} \circ g)|_V$$

But $\tilde{f} \circ g$ is smooth so $(f \circ g)|_V$ is smooth. //

If $\gamma : (-\varepsilon, \varepsilon) \to S$ is a path through s in S ~~then~~ & $f : S \to \mathbb{R}^M$ is smooth we have $f \circ \gamma$ is smooth.

Prop^n 3.14

If $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to S$ & $f : S \to \mathbb{R}^m$ is smooth ~~then~~ if $\gamma_1'(0) = \gamma_2'(0) \in \mathbb{R}^N$ we have

$$(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0).$$

Proof ~~...~~ Extend f to $\tilde{f}$ on a nbd of $\gamma_1(0) = \gamma_2(0) = s \in S$. Then

$$(f \circ \gamma_1)'(0) = (\tilde{f} \circ \gamma_1)'(0) = \tilde{f}'(s)(\gamma_1'(0))$$
$$(f \circ \gamma_2)'(0) = \quad \cdot \quad \cdot \quad = \tilde{f}'(s)(\gamma_2'(0))$$

//

It follows from this Prop'n that if $v \in T_s S$ we can define $f'(s)(v)$ by choose $\gamma : (-\varepsilon, \varepsilon) \to S$ s.t. $\gamma'(0) = v$ & letting $f'(s)(v) = (f \circ \gamma)'(0)$. This is well-defined i.e. if $\tilde{\gamma}'(0) = v$ get same answer

Prop^n 3.15 The map $f'(s) : T_s S \to \mathbb{R}^m$ is linear.

Proof. ~~...~~ Extend f to $\tilde{f}$ then $f'(s)(v) = \tilde{f}'(s)(v)$ from the proof of the prop. Hence

$$f'(s) = \tilde{f}'(s)\big|_{T_s S} \text{ is linear //}$$

Prop^n 3.16

Let $S \subseteq \mathbb{R}^N$ a submanifold & $f : S \to \mathbb{R}^M$ smooth. If $g : U \to \mathbb{R}^N$, $g(U) \subseteq S$ & is smooth then

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

Proof Extend f to $\tilde{f}$ on a nbd of $g(x)$. Then $f \circ g = \tilde{f} \circ g$ and

$$(f \circ g)'(x) = (\tilde{f} \circ g)'(x)$$
$$= \tilde{f}'(g(x))\, g'(x)$$

But $f'(g(x)) \doteq \tilde{f}'(g(x))\big|_{T_{g(x)} S}$

& $g'(x)$ has image in $T_{g(x)} S$

$\therefore$ $(f \circ g)'(x) = f'(g(x))\, g'(x)$. //

§4. Geometry of curves

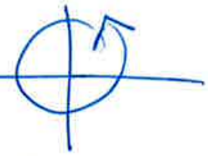
Def^n 4.1 A curve $C \subseteq \mathbb{R}^N$ is a 1-dim^{l} submanifold.

If C is a curve & $c \in C$ ~~then~~ then $T_c C$ is 1-dim^{l} & $T_c C - \{0\}$ has two pieces ie connected compone

Call a choice of one of these an orientation for $T_c C$. If we make this choice continuously as c varies ~~as co~~ for all c we say C is oriented. Call a curve with an orientation an oriented curve.

E.G. ① $F(x,y,z) = x^2+y^2-1$ $\quad C = S^1$.

~~$f(x,y,z) = (\cos z, \sin z, z)$~~



② $F(x,y,z) = (x - \cos z,\ y - \sin z)$

$F : \mathbb{R}^3 \to \mathbb{R}^2$

$$F'(x,y,z) = \begin{pmatrix} 1 & 0 & +\sin z \\ 0 & 1 & -\cos z \end{pmatrix}$$

onto $\therefore$ $F(x,y,z)=0$ is a curve

16·8

If C is an oriented curve & locally C has a parametrisation $\gamma : (a,b) \to C$ and if $\gamma'(t)$ is in the oriented half of $T_{\gamma(t)} C$ we say γ is oriented.

Defn 4.2 A ~~oriented~~ parametrised curve is a curve C for which there is a parametrisation

$$\gamma : (a,b) \to C$$

with $\gamma(a,b) = C$.

EG: ① Can't do it.

Proof Assume that $\gamma(a,b) \simeq S^1$.

Remove t $\quad$ $\gamma(t)$

Two pieces $\quad$ 1-piece.

② $\gamma(t) = (\cos(t), \sin(t), t)$

Notice that Prop 3.12 tells us that if

$\gamma : (a,b) \to C$ & $\mu : (c,d) \to C$

are two parametrisations & $\gamma(a,b) = C = \mu(c,d)$

then $\exists$ $h : (a,b) \to (c,d)$ a diffeo^m

s.t. $\mu \circ h = \gamma$.

& there are lots of parametrisations.

One way to fix one is to parametrise by

arc-length.

If $t_0 < t \in (a,b)$ the distance

along the curve from $\gamma(t_0)$ to $\gamma(t)$ is

$$\int_{t_0}^{t} \|\gamma'(\xi)\| \, d\xi .$$

Assume $$\int_{t_0}^{t} \|\gamma'(t)\| dt = t - t_0$$

$\forall t, t_0$. Differentiating both sides gives

$\|\gamma'(t)\| = 1$

Def^n 4.3 If $\gamma : (a,b) \to C$ is a parametrisation of C

we say it is a parametrisation by arc length if $\|\gamma'(t)\| = 1$ $\forall t \in (a,b)$

End