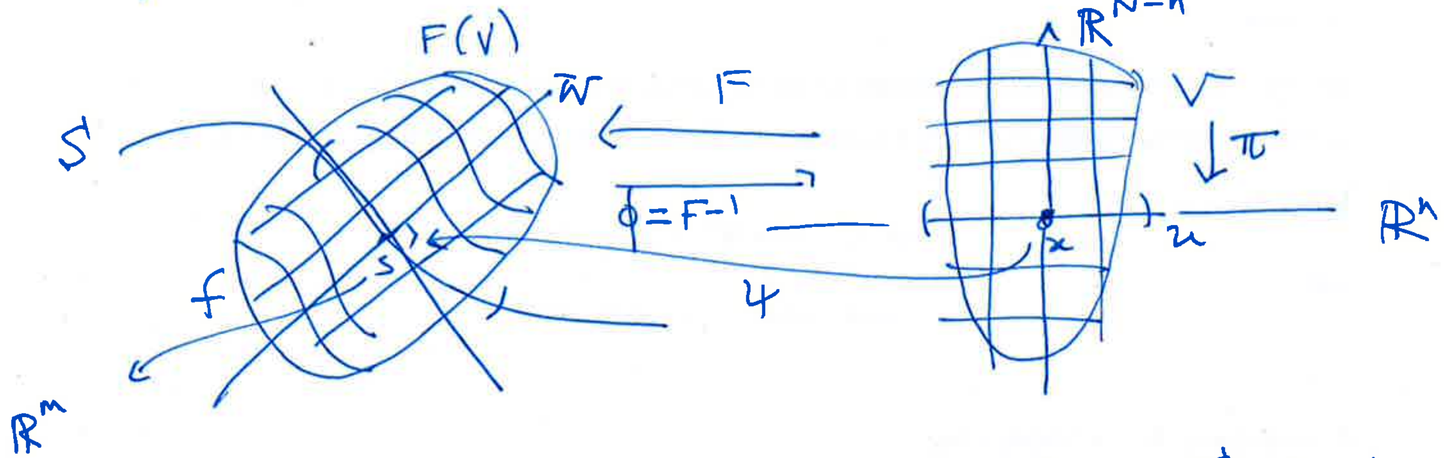
Review last proof

$\psi : \mathcal{U} \longrightarrow S$ $\qquad\qquad$ $f : S \longrightarrow \mathbb{R}^m$

parametrisation



$$F(x,y) = \psi(x) + \sum y^i w_i$$

$$\pi(x,y) = x$$

$\psi \circ \pi \circ \phi \quad F(V) \longrightarrow F(V) \cap S$

If $\quad x \in S \qquad x = \psi(u)$

$\qquad\qquad\qquad\qquad\qquad\quad = F(u, 0).$

$$\phi(x) = (u, 0)$$

$$\pi\, \phi(x) = u.$$

$$\psi\, \pi\, \phi(x) = \psi(u) = x.$$

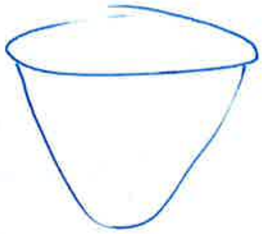
$\therefore \quad \psi\pi\phi(x) = x \quad \text{if} \quad x \in F(V) \cap S$

$\tilde{f} = f \circ (\psi \circ \pi \circ \phi) = (f \circ \psi) \circ \pi \circ \phi$

smooth

$\underline{\tilde{f}(x) = f(x)}.$

(Lecture 16)

Examples (1) P  16.2

$F(x,y,z) = z - (x^2+y^2)$.

$h(x,y,z) = z$ is smooth.

as $\tilde{h}: \mathbb{R}^3 \to \mathbb{R}$ $\quad \tilde{h}(x,y,z) = z$ is smooth

& h is $\tilde{h}|_P$

or if $\varphi(x,y) = (x, y, x^2+y^2)$.

$$f \circ \varphi(x,y) = x^2+y^2$$

is smooth.

(2) S^2: $\quad$ ~~$f(x,y,z) =$~~

$$h(x,y,z) = z \cdot x^2 + y$$

smooth as $\tilde{h}(x,y,z) = zx^2 + y$ smooth on $\mathbb{R}^3$

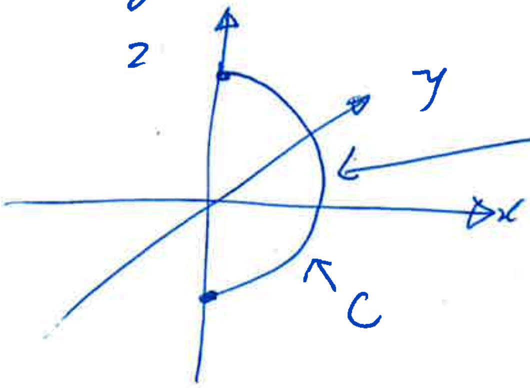
& $h = \tilde{h}|_{S^2}$.

Or parameter: $\varphi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi)$

$$h \circ \varphi = \cos\theta \cos^2\theta \sin^2\phi + \sin\theta \sin\phi$$

smooth.

Note image of Ψ not all of S^2



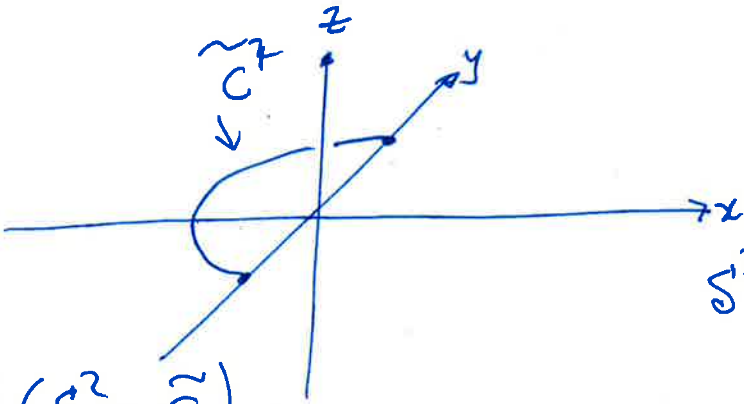
H is just $S^2 - C$

Try $\tilde{\Psi}(\theta, \phi) = (-\cos\theta\sin\phi,\ \cos\phi,\ \sin\theta\sin\phi)$

we miss : $(-\sin\phi,\ \cos\phi,\ 0)$

$\underline{\underline{\theta = 0, 2\pi}}$

$\phi = 0, \pi$



$S^2 - \tilde{C}$

$S^2 = (S^2 - C) \cup (S^2 - \tilde{C})$

These parameter cover all of S^2.

$$h \circ \tilde{\Psi}(\theta, \phi) = -\sin\theta\sin^3\phi\cos\theta + \cos\phi$$

Smooth on $(0, 2\pi) \times (0, \pi)$

Easier to use the restriction if you can!

Propⁿ B.12

Let $S \subseteq \mathbb{R}^N$ be a submanifold.
If $\psi : \mathcal{U} \to S$ & $\chi : V \to S$ are local parametrisations with $\psi(\mathcal{U}) = \chi(V)$ then

Th: $\chi^{-1} \circ \psi : \mathcal{U} \to V$ is a diffeoⁿ.
(SHOES & SOCKS)

Proof Clearly $(\chi^{-1} \circ \psi)^{-1} = \psi^{-1} \circ \chi$ so by symmetry it suffices to show that $\chi^{-1} \circ \psi$ is smooth..

But $\chi^{-1} : \chi(V) \longrightarrow V$ is smooth because $\chi^{-1} \circ \chi : V \to V$ is the identity.
But then $\chi^{-1} \circ \psi$ is smooth //

If $f : S \to \mathbb{R}^m$ is smooth & $s \in S$ let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be a smooth path in $\mathbb{R}$ through s. By extending f to a smooth fⁿ $\tilde{f}$ in a nbd $\mathcal{U}$ of s and possibly shrinking $(-\varepsilon, \varepsilon)$ we can arrange $\gamma(-\varepsilon, \varepsilon) \subseteq \mathcal{U}$ so
$f \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^m$
is equal to $\tilde{f} \circ \gamma$ which is C^∞.

Define $f'(s) : T_s S \to \mathbb{R}^M$

$\cancel{f'(s)(\gamma'}$

by $$f'(s)(v) = \frac{d}{dt}\left(\tilde{f}\circ\gamma(t)\right)\Big|_{t=0}$$

where $\gamma'(t) = v$.

Note that

$$f'(s)(v) = \frac{d}{dt}\left(\tilde{f}\circ\gamma(t)\right)\Big|_{t=0}$$
$$= \tilde{f}'(s)(\gamma'(0))$$
$$= \tilde{f}'(s)(v).$$

~~Hence~~ Notice that $\tilde{f}\circ\gamma(t) = f\circ\gamma(t)$ so derivative is independent of choice of $\tilde{f}$.

<u>Prop^n 3.13</u> If $f : S \to \mathbb{R}^m$ then

$f'(s)\, T_sS \to \mathbb{R}^m$ is ~~linear~~. ~~linea~~ well-defined and linear.

For any extension $\tilde{f}$ to an open set

containing $\mathcal{U}$ we have

$$f'(s) = \tilde{f}'(s)\Big|_{T_s S}$$

$$\left(\tilde{f}'(s) : \mathbb{R}^N \to \mathbb{R}^M\right.$$
$$\cup\!|$$
$$T_s S$$

↓ End 16.