$\Psi: U \to S$

parametrisation

\[ F: S \to \mathbb{R}^n \]

$\phi = F^{-1}$

\[ \pi(x, y) = x \]

$F \circ \pi \circ \phi : F(V) \to F(V) \cap S$

If $x \in S$, then $x = \Psi(u) = F(u \setminus o)$. 

\[ \phi(x) = (u, o) \]

$\pi \circ \phi(x) = u$. 

$\Psi \circ \phi(x) = \Psi(u) = x$. 

$\Psi \circ \pi \circ \phi(x) = x \quad \forall x \in F(V) \cap S$ 

$\tilde{f} = f \circ (\Psi \circ \pi \circ \phi) = (f \circ \Psi) \circ \pi \circ \phi$

$\tilde{f}(x) = f(x)$.
$F(x, y, z) = z - (x^2 + y^2)$.

$h \Phi(x, y, z) = z$ is smooth.

as $\tilde{h} : \mathbb{R}^3 \to \mathbb{R}$

$h(x, y, z) = z$ is smooth

$a h$ is $\tilde{h} |_p$
n
or $\tilde{h}$.

If $\psi(x, y) = (x, y, x^2 + y^2)$, then

$f \circ \psi(x, y) = x^2 + y^2$

is smooth.

(2) $\mathbb{S}^2$

$f(x, y, z) = x^2 + y^2$

$h(x, y, z) = z - x^2 - y$

smooth and $\tilde{h}(x, y, z) = z - x^2 + y$ smooth on $\mathbb{R}^3$

$h = \tilde{h} |_{\mathbb{S}^2}$.

Or parameter: $\psi(0, \theta) = (\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi)$

$h \circ \psi = \cos \phi \cos \theta \sin^2 \phi + \sin \phi \sin \theta$

smooth.
Note image at \( \theta \) not all of \( S^2 \)

Then \( \widetilde{\psi}(0, \phi) = (-\cos \phi \sin \phi, \cos \phi, \sin \phi \cos \phi) \)

we miss : \((-\sin \phi, \cos \phi, 0)\)

\[ \begin{align*}
\phi &= 0, \pi \\
\theta &= 0, 2\pi
\end{align*} \]

\[ S^2 = (S^2 - C) \cup (S^2 - \tilde{C}) \]

These parameters cover all of \( S^2 \):

\[ h \circ \widetilde{\psi}(0, \phi) = -\sin^3 \phi \cos \phi + \cos \phi \]

Smooth on \( (0, 2\pi) \times (0, \pi) \)

Easier to use the restriction if you can!
Prop 8.12

Let $S \subseteq \mathbb{R}^n$ be a submanifold. If $\psi: U \to S$ & $\chi: V \to S$ are local parametric data with $\psi(U) = \chi(V)$ then $\chi^{-1} \circ \psi: U \to V$ is a diffeomorphism $(\text{shoes & socks})$

Proof: Clearly $(\chi^{-1} \circ \psi)^{-1} = \psi^{-1} \circ \chi$ so by symmetry it suffices to show that $\chi^{-1} \circ \psi$ is smooth.

But $\chi^{-1}: \chi(V) \to V$ is smooth because $\chi^{-1} \circ \chi: V \to V$ is the identity.

But then $\chi^{-1} \circ \psi$ is smooth.

If $f: S \to \mathbb{R}^m$ is smooth & set $s: (-\varepsilon, \varepsilon) \to \mathbb{R}$ be a smooth path in $\mathbb{R}$ through $s$. By extending $f$ to a smooth $f^\sim: \mathbb{R}^m$ in a neighborhood of $s$ and possibly shrinking $(-\varepsilon, \varepsilon)$ we can arrange for $s: (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ is equal to $\frac{\partial f}{\partial y}$ which is $C^0$.  

Define \( f'(s) : T_s S \to \mathbb{R}^m \)

\[
f'(s)(v) = \frac{d}{dt} \left. \tilde{f}^* \circ \gamma(t) \right|_{t=0}
\]

where \( \gamma'(t) = v \).

Note that

\[
f'(s)(v) = \frac{d}{dt} \left. \tilde{f}^* \circ \gamma(t) \right|_{t=0}
= \tilde{f}'_{\tilde{T} u} f'(s)(\gamma'(0))
= \tilde{f}'_{\tilde{T} u} f'(s)(v).
\]

Notice that \( f \circ \gamma(t) = f \circ \tilde{\gamma}(t) \) so the derivative is independent of choice of \( \tilde{f} \).

Prop. 3.13 If \( f : S \to \mathbb{R}^n \) then

\( f'(s)(T_s S) \to \mathbb{R}^m \) is well-defined and \( \tilde{T} \)-linear.

For any extension \( f_{\tilde{T} u} \) to an open set containing \( U \) we have

\[
f'(s) = \tilde{f}'_{\tilde{T} u} f'(s) \bigg|_{T_s S}
\]

( \( f'(s) : \mathbb{R}^n \to \mathbb{R}^m \))

End 16.