Lecture 15

Test 21 Sept

Probably 1Q – T/F short answer

1Q – "calculation"

Recall

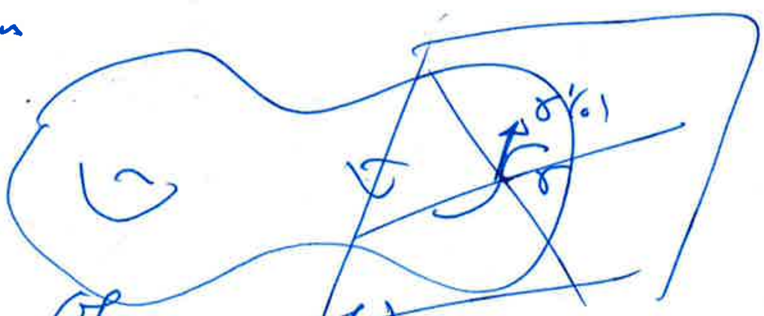
$S \subseteq \mathbb{R}^N$

$T_s S = \{ \gamma'(0) \mid \gamma : \text{a path in } S \text{ through } s \}$

$= \operatorname{im} f'(x)$ f a parametrisation

$= \ker F'(x)$ F a defining ~~function~~ equation

= span



Spanned by $\left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right)$

Another example of a tangent space

$$F(x,y,z) = z - (x^2+y^2) \qquad F: \mathbb{R}^3 \to \mathbb{R}$$

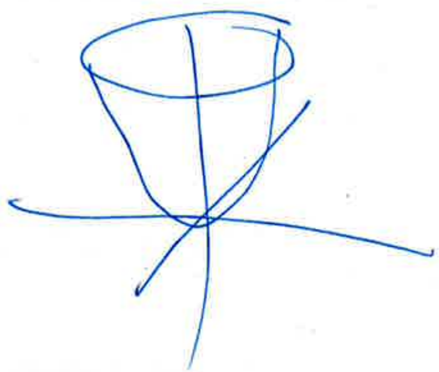
~~$F(x,y,z) =$~~

$$\left.\begin{aligned} \frac{\partial F}{\partial z} &= 1 \\ \frac{\partial F}{\partial x} &= -2x \\ \frac{\partial F}{\partial y} &= -2y \end{aligned}\right\} \text{ never zero on } \mathbb{R}^3$$

$$\therefore P = F^{-1}(0,0,0) = \{(x,y,z) \mid z = x^2+y^2\}$$

$=$ paraboloid

is a submanifold.



$$T_{(x,y,z)}P = \{(\alpha,\beta,\gamma) \mid \cancel{\alpha - 2x\beta - 2y\gamma = 0}\}$$

$$-2x\alpha - 2y\beta + \gamma = 0$$

$=$ plane

Parametrization

$$f(x,y) = (x,\ y,\ x^2+y^2)$$

1-1 $\quad f'(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x & 2y \end{pmatrix} \quad$ 1-1

$V = \mathbb{R}^3$

~~$M = \mathbb{R}^2$, $V = \{(x,y,z) \mid z > 0\}$~~

P ~~$P \cap V = P =$~~

$$\operatorname{Im} f'(x,y) = \operatorname{span}\left(\begin{pmatrix}1\\0\\2x\end{pmatrix}, \begin{pmatrix}0\\1\\2y\end{pmatrix}\right)$$

~~ker~~ Both in ker $F'(x,y)$. & ~~to~~ linearly ind.

This example is nice because P is the graph of $h(x,y) = x^2 + y^2$.

Want to talk about functions on a submanifold. Need to ~~sa~~ be careful about the difference between a function and its restriction

eg $\quad f : \mathbb{R} \to \mathbb{R} \qquad f(x) = \sin(x)$

$\quad g : (0, \pi) \to \mathbb{R} \qquad g(x) = \sin(x)$

f is not 1-1 $\qquad$ usually call g

g is 1-1 $\qquad$ $f|_{(0,\pi)}$.

Restricting functions

If $f : X \to Y$ is a function & $Z \subseteq X$ we denote by $f|_Z : Z \to Y$ the <u>restriction of f to Z</u>. Often it is important to distinguish between f & $f|_Z$. ~~For example $f|_Z$ may be invertible when f is not.~~

We have the following useful results

(1) If $f : \mathcal{U} \to \mathbb{R}^m$ is C^k for $\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n$ & $V \underset{\text{open}}{\subseteq} \mathbb{R}^n$ $V \subseteq \mathcal{U}$ then $f|_V : V \to \mathbb{R}^m$ is C^k.

(2) If $f : \mathcal{U} \to \mathbb{R}^m$ & $\mathcal{U} = \bigcup_{\alpha \in I} V_\alpha$ $V_\alpha \subseteq \mathbb{R}^n$ open & $f|_{\mathcal{U}_\alpha}$ is C^k then f is C^k.

Because f is C^k on $\mathcal{U}$ $\iff$ f is C^k on an open nbd of every $x \in \mathcal{U}$.

(Start calling parametrisations ψ) 15.5 ~~4.3~~

~~Talk about S^2 $(\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi)$ Parameters~~

Smooth functions on submanifolds

Let $S \subseteq \mathbb{R}^N$ be a submanifold.

Def^n 3.10 We say that $f : S \to \mathbb{R}^n$ is smooth if $\forall s \in S$ $\exists\, \mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^N$ & $\tilde{f} : \mathcal{U} \to \mathbb{R}^n$ smooth such that $f/_{S\cap\mathcal{U}} = \tilde{f}/_{S\cap\mathcal{U}}$

(i.e $f(x) = \tilde{f}(x)$ $\forall x \in S\cap\mathcal{U}$).

Note ① We say that f extends locally to a smooth function on a neighbourhood of any point.

② $\tilde{f}$ is not unique.

③ If $f : \mathbb{R}^N \to \mathbb{R}^m$ is C^∞ then $f/_S$ is smooth. Typically this is what we want.

~~$(-1,1)\times(-1,1) = \mathcal{U}$, $S = \{(0,y)\}$, $f(y) = y^2$, $\tilde{f}(x,y) = y^2$ with bump or $\tilde{f}_U(x,y) = \varphi(x)y^2$~~

Prop^n 3.11 Let S be a submanifold & $f : S \to \mathbb{R}^n$ a function.

(1) If $\psi : \mathcal{U} \to S$ ($\mathcal{U} \subseteq_{\text{open}} \mathbb{R}^n$) is a parametrisation & f is smooth then $f\circ\psi$ is smooth

(2) If for every $s \in S$ there is a parametrisation $\psi : \mathcal{U} \to S$ ($\mathcal{U} \subseteq_{\text{open}} \mathbb{R}^n$) with $s \in f(\mathcal{U})$ & $f\circ\psi$ smooth then f is smooth.

<u>Proof</u>

① Let $x \in \mathcal{U}$ than $\psi(x) \in S$. Chose V open in $\mathbb{R}^N$ s.t. $V \ni \psi(x)$ & $\exists$ $\tilde{f} : V \to \mathbb{R}^n$ smooth & $\tilde{f}/_{S \cap V} = f/_{S \cap V}$.

~~As $\psi(y) \in S \cap V$~~

Let $W = \psi^{-1}(V)$ open in $\mathcal{U}$ & containing x. Then $\psi(W) \subseteq S \cap V$ so

$$f \circ \psi/_W = \tilde{f} \circ \psi/_W$$ smooth by chain rule.

NOT "f"

Hence we can cover $\mathcal{U}$ by open sets W so that $f \circ \psi$ on each one is smooth

$\therefore f \circ \psi$ is smooth.

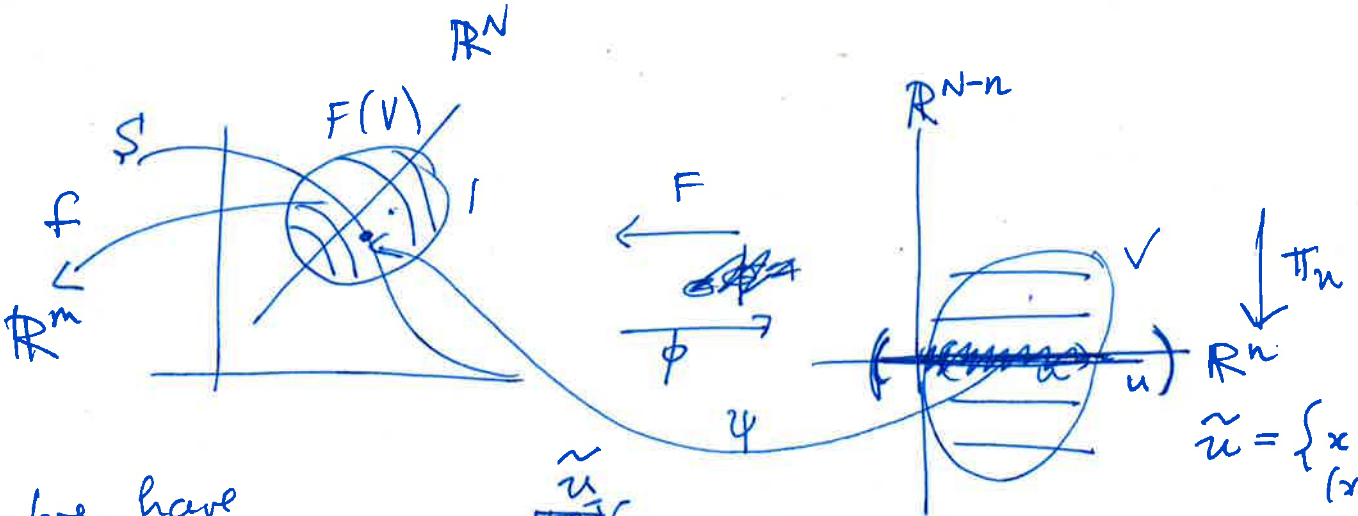
② Let $s \in S$ and chose a parametrisation $\psi : \mathcal{U} \to S$ with $s = \psi(x)$ & $f \circ \psi$ smooth. Recall from the proof of Th^m 3.2 that if we let $W = \operatorname{im} \psi'(x)^\perp$ & $w^1, \ldots, w^n$ a basis of W we define

$$F(x, y) = \psi(x) + \sum_i w^i y^i$$

& in a nbd of $(x, 0)$ & $(s) = F(x, 0)$

F is a diffeo^m. from V to $F(V)$ say

Let $\phi : F(V) \to V$ be F^{-1}. Then



$\tilde{u} = \{x \in u \mid (x,0) \in V\}$

We have ~~Consider~~ $f \circ \psi : \tilde{u} \to \mathbb{R}^m$ which is C^∞ by assumption. Consider

$\tilde{f}|_{F(V)} : F(V) \longrightarrow \mathbb{R}^m$

$$= \underbrace{f \circ \psi}_{C^\infty} \circ \underbrace{\pi_u}_{C^\infty} \circ \underbrace{\phi}_{C^\infty}$$

This is C^∞ as $f \circ \psi$ is C^∞ & π_u is C^∞

~~$x \in S \cap F(V)$ then $F(x)$ $\phi(x) =$~~

where $\pi_u(x,y) = x$. If $x \in S$ then

$\phi(x) = (u, 0)$ $\quad$ ($\pi_u(\phi(x)) = \pi_u(u,0) = u$

$x = F\phi(x) = \psi(u)$. From previous page

$\therefore f \circ \psi \circ \pi_u \circ \phi(x) = f \circ \psi(u) = f(x)$

$\therefore$ have extension. //