# Lecture 14

**Recall**

submanifolds
- def
- local param
- local defining eqⁿ
- locally a graph

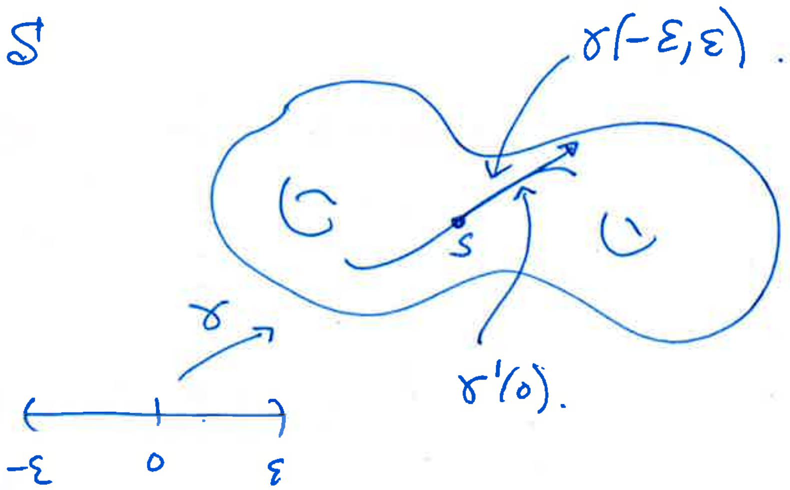
**Tangent space**

$\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^N$ smooth

$\gamma(-\varepsilon, \varepsilon) \subseteq S$

$\gamma(0) = s$

$\gamma'(0) \in \mathbb{R}^N$




~~Define~~:

Previously denoted by $\gamma'(0)$ the linear map $\gamma'(0) : \mathbb{R} \to \mathbb{R}^N$. We will abuse notation and also denote ~~to~~

small enough we can assume $\gamma(-\varepsilon,\varepsilon)\subseteq \mathcal{U}$

Then $\cancel{\phi}\psi\circ\gamma(t)=0 \quad \forall t\in(-\varepsilon,\varepsilon)$. $\therefore (\psi\circ\gamma)'(0)=0$. By the

Chain rule $\psi'(s)(\gamma'(0))=0$.

$\therefore \quad \gamma'(0)\in \ker\psi'(s)$

ie $\boxed{T_s S \subseteq \ker\psi'(s).}$

Assume $v\in\ker\psi'(s)$

Let $\gamma_v(t) = \phi^{-1}(\phi(s)+t\,\phi'(s)(v))$

$\gamma_v:(-\varepsilon,\varepsilon)\longrightarrow \mathbb{R}^{N}$ $\qquad \gamma_v(0)=s.$

for some suitable $\varepsilon$.

$$\gamma_v'(0) = (\phi^{-1})'(\phi(s))\,(\phi'(s)(v)) \quad \text{(Chain rule).}$$

$$= (\phi^{-1}\circ\phi)'(s)(v)$$

$$= \cancel{(\phi'(s))^{-1}\,\phi'(s)(v)}$$

$$= v$$

But $\phi^{n+i}(\gamma_v(t)) = (\psi^i(s) + t(\phi'(s)(v))^i)$ if $1\le i\le N-n$ ~~if $n+1\le i\le N$~~

$\phi^{n+i}(\gamma_v(t)) = \psi^i(\gamma_v(t))$, $\psi^i(s)=0$, $(\phi'(s)(v))^i = \psi'(s)(v)^i = 0.$

$v\in\ker\psi'(s)$

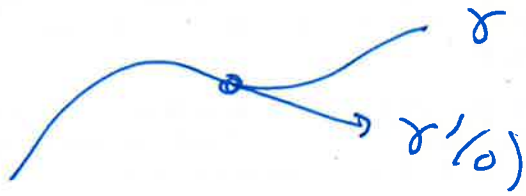
$\therefore \gamma_v(t)\in S$

So $\gamma_v(t)$ is a path $\therefore v=\gamma_v'(0)\in T_s S$

$\therefore \ker\psi'(s) = T_s S$ //

vector $\left.\frac{d\gamma}{dt}\right|_{t=0} = \lim_{\xi \to 0} \frac{\gamma(\xi) - \gamma(0)}{\xi}$ ~~13.8~~ 14.2

by $\gamma'(0)$




<u>Def$^n$ 3.7</u> Denote by $T_s S$ the union of all the vectors $\gamma'(0)$ for $\gamma$ a smooth path in $S$ through $s$. Call it the tangent space to $S$ at $s$.

<u>Prop$^n$ 3.8</u> $T_s S$ is an ~~a~~ $n$-dim$^l$ subspace of $\mathbb{R}^N$.

<u>Proof</u> Choose an open $U \subseteq \mathbb{R}^N$ with $s \in S$, $\phi : U \xrightarrow{\text{dif}} \mathbb{R}^N$, $\phi(U) \underset{\text{open}}{\subseteq} \mathbb{R}^N$ & $\phi$ a diffeo$^m$

$$S \cap U = \{ x \mid (\phi^{n+1}(x), \dots, \phi^N(x)) = 0 \}$$

Let $\psi : U \to \mathbb{R}^{N-n}$. $\psi(x) = (\phi^{n+1}(x), \dots, \phi^N(x))$

We show that $T_s S = \ker \psi'(s)$.

First let $\gamma : (-\varepsilon, \varepsilon) \to S$ be a smooth path in $S$ through $s$. <u>By choice $\varepsilon$</u>

(EXPAND ON THIS)

TALK ABOUT CHOICE OF DEFIN | ~~13.10~~ 14.4

**Propn 8.9** Let $S$ be a submanifold & $s \in S$.

Then

① If $F : \mathcal{U} \to \mathbb{R}^{N-n}$ is a local defining equation for $S$ & $s \in \mathcal{U}$ then

$$T_s S = \ker F'(s)$$

② If $f : \underset{\mathbb{R}^n}{\overset{U}{\cap\!\!|}} \to \underset{\text{open}}{\mathcal{U}} \subseteq \mathbb{R}^N$ is a local parametrisation for $S$ & $f(x) = s$ then

$$T_s S = \operatorname{im} f'(x)$$

Moreover $T_s S$ is spanned by the columns of $f'(x)$ ie the vectors

$$f'(x)(e^1) \ldots f'(x)(e^n) \quad \text{or}$$

$$\frac{\partial f}{\partial x^1}(x), \ldots, \frac{\partial f}{\partial x^n}(x).$$

**Proof**

① If ~~...~~ $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^N$ is a path in $S$ through $s$ by changing $\varepsilon$ can assume $\gamma(-\varepsilon, \varepsilon) \subseteq \mathcal{U} \cap S$

$\therefore F \circ \gamma = 0 \therefore (F \circ \gamma)'(0) = 0$

$\therefore F'(\gamma(0))(\gamma'(0)) = 0$

~~13.4~~ 14.5

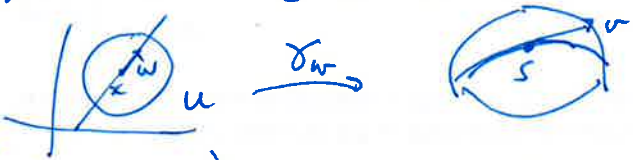
$\therefore \quad \gamma'(0) \in \ker F'(s)$

$\therefore \quad T_s S \subseteq \ker F'(s)$

But $F'(s) : \mathbb{R}^N \to \mathbb{R}^{N-n}$ is onto so kernel is $n$ dim$^l$ $\therefore$ $T_s S = \ker F'(s)$
(Need to know $T_s S$ is a <u>subspace</u> & of dim$^n$ $n$).

(2) Let $v \in \operatorname{im} f'(x)$. $\therefore$ $\exists\, w \in \mathbb{R}^n$ &

$v = f'(x)(w)$.




Define $\gamma_w(t) = f(x + tw)$ $\quad \gamma_w(0) = f(x) = s$

This is a path in $S$ through $s$

$\gamma_w'(0) = f'(x)\left(\frac{d}{ds}(\cancel{sx}\ x + sw)\Big|_{s=0}\right)$

$= f'(x)(w) = v.$

~~Agree~~ $\therefore$ $\boxed{\operatorname{im} f'(x) \subseteq T_s S}$

& $f'(x)$ is 1-1 $\therefore$ $\dim \operatorname{im} f'(x) = n$ $\therefore$ equal

As $e^1 \ldots e^n$ are linearly independent & $f'(x)$ is 1-1 we have $f'(x)(e^1) \ldots, f'(x)(e^n)$ linearly independent (<u>Ex</u>). Hence $\operatorname{im} f'(x)$ is $n$ dim$^l$ $\therefore$ $\operatorname{im} f'(x) = T_s S$

& $\frac{\partial f}{\partial x^1}(x), \ldots, \frac{\partial f}{\partial x^n}(x)$ are a basis of $T_s S$

## Example $S^2$

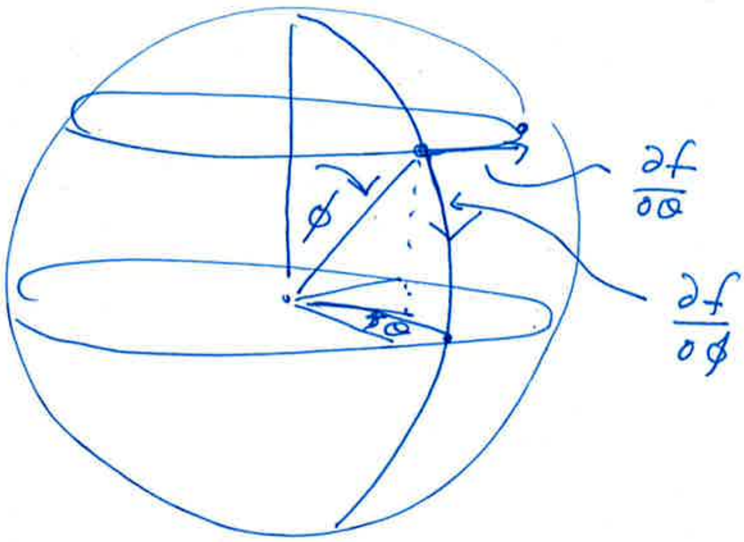
We saw in Ass 2 that if

$$f(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi)$$

$$f : (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$$

then
$$\frac{\partial f}{\partial \theta} = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$$

$$\frac{\partial f}{\partial \phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$$




Note
$$\left\| \frac{\partial f}{\partial \theta} \right\|^2 = \sin^2\phi$$

$$\left\| \frac{\partial f}{\partial \phi} \right\|^2 = 1.$$