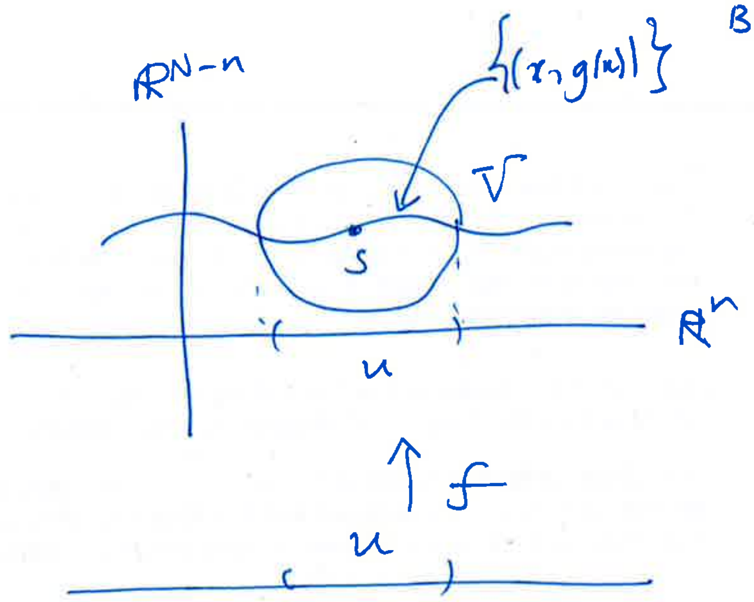
Lecture 13

(3) $\Rightarrow$ (4)



For simplicity assume there is no need to permute indices.

$\exists\, V$ open $\subseteq \mathbb{R}^N$

$s \in V$

& $S \cap V = \{ (x, g(x)) \mid x \in U \}$ $\quad g : U \to \mathbb{R}^{N-n}$

Let $f(x) = (x, g(x))$ $\quad f : U \longrightarrow \mathbb{R}^N$

f is 1-1, $f'(x) = (I, g'(x))$ is 1-1

& $f(U) = \{ (x, g(x)) \mid x \in U \} = S \cap V$.

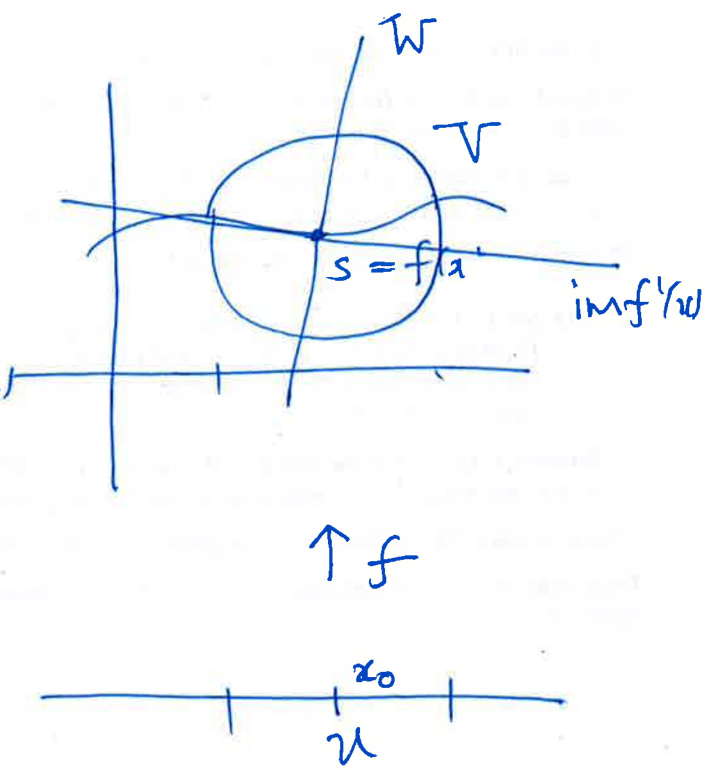
(4) $\Rightarrow$ (1)

Let $s \in S$, V open in $\mathbb{R}^N$ containing s.

$f : U \to \mathbb{R}^N$, f 1-1 f' 1-1

$f(U) = S \cap V$

Choose x_0 s.t. $f(x_0) = s$.



$\operatorname{im} f'(x_0)$ is n-dim$^{\ell}$ ($f'(x_0)$ is 1-1)

~~$\operatorname{im} f'$~~ $\therefore\; W = (\operatorname{im} f'(x_0))^{\perp}$ is $N-n$ dim$^{\ell}$.

Pick a basis $w^1, \dots, w^{N-n}$.

Define $\quad F : \mathcal{U} \times \mathbb{R}^{N-n} \longrightarrow \mathbb{R}^N$

$$F(x,y) = f(x) + \sum_{i=1}^{N-n} w_i y^i$$

$$\therefore\; F(x_0, 0) = f(x_0) = s$$

Let $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^{N-n}$

$$F'(x_0, 0)(\alpha, \beta) = \underbrace{f'(x_0)(\alpha)}_{\in\, \operatorname{Im} f'(x_0)} + \underbrace{\sum_{i=1}^{N-n} w_i \beta_i}_{\in\, \operatorname{Im} f'(x_0)^{\perp}}$$

$$\therefore\; F'(x_0,0)(\alpha,\beta) = 0 \iff \begin{cases} f'(x_0)(\alpha) = 0 & \iff \alpha = 0 \\ \sum_{i=1}^{N-n} w_i \beta_i = 0 & \iff \beta = 0 \end{cases}$$

$\therefore\; F'(x_0, 0)$ is 1-1. Also square so

$F'(x_0, 0)$ is invertible so locally a

diffeo^{m} by IF Th^{m}.

so $\exists\, \tilde{V} \underset{\text{open}}{\subseteq} \mathcal{U} \times \mathbb{R}^{N-n}$ s.t. $(x_0, 0) \in \tilde{V}$

s.t. $\quad F : \tilde{V} \longrightarrow F(\tilde{V}) \underset{\text{open}}{\subseteq} \mathbb{R}^N$ is a diffeo^{m}

Let $\bar{V} = F^{-1}(V) \cap \tilde{V}$. open .

Then $F : \bar{V} \longrightarrow F(\bar{V}) \underset{\text{open}}{\subseteq} \mathbb{R}^N$ is also a diffeo

$$\begin{array}{c} \cap \! \mid \\ V \end{array}$$

Let $\phi : F(\bar{V}) \to \bar{V}$ be the inverse of F.

Let $z \in F(\bar{V}) \cap S$ then

$$z = f(x) = F(x, 0)$$

$$\therefore \quad \phi(z) = (x, 0)$$

f ~~Let (x, 0~~ Also let $z \in F(\bar{V})$ & $\phi(z) = (x, 0)$

Then $z = F(x, 0) = f(x) \quad \therefore \quad z \in S$.

$$\therefore \quad S \cap F(\bar{V}) = \{ z \mid \phi^{n+1}(z) = \cdots = \phi^N(z) \}$$

Hence $F'(x_0, 0)$ is invertible by dimension count.

Let ϕ be a local inverse for F. If $z \in S$ ~~$(u,v) \in S$~~ ~~then~~ $z = F(x,0)$

~~If $(x,y) \in S$~~ ~~$(x,y) = f(u) = F(y,0)$~~

$$\phi(z) = \phi F(x,0) = (x,0)$$

$$\therefore \phi^{n+1}(u) = \dots = \phi^{N}(z) = 0$$

& if $\phi^{n+1}(u) = \dots = 0$ $\phi(u) = (x,0)$ $u = F(x,0) \in S$. //

Corollary 3.3

If $U \underset{\text{open}}{\subseteq} \mathbb{R}^N$ & $F : U \to \mathbb{R}^{N-n}$ is smooth with $F'(x)$ onto $\forall x \in S = F^{-1}(0)$ then S is a submanifold of dim^n n.

~~Note Corollary 3.2 ... if $f : \mathbb{R}^n \to \mathbb{R}^{N-n}$ is a smooth function the graph of f is a submanifold~~

Defn 3.4

Let $S \subseteq \mathbb{R}^N$ be a submanifold. If $U \subseteq \mathbb{R}^N$, U open & $F : U \to \mathbb{R}^{N-n}$ smooth $F'(s)$ onto $\forall s \in U \cap S$ & $S \cap U = \{s \in U \mid F(s) = 0\}$ then F is called a <u>local defining eq^s</u> for S

Defn 3.5

Let $S \subseteq \mathbb{R}^N$ be a submanifold. If $U \subseteq \mathbb{R}^n$ open, $V \subseteq \mathbb{R}^N$ open &

$f : U \to V$ is smooth, 1-1, $f'(x)$ is 1-1 $\forall x \in U$ & $f(U) = V \cap S$ then f is called a <u>local parametrisation of S</u>

<u>Notice</u> ① Neither of these are unique

② If $N = 3$, $n = 2$ $F : U \subseteq \mathbb{R}^3 \to \mathbb{R}$

$F^{-1}(0)$ is a submanifold if $F'(x) \neq 0$ $\forall x \in F^{-1}(0)$.

13.5

Example If $f : \underset{\text{open}}{U} \subseteq \mathbb{R}^n \to \mathbb{R}^{N-n}$ smooth

then $\text{graph}(f) = \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{R}^{N-n}$

is a submanifold of dim^n n

This follows from Theorem 3.2.

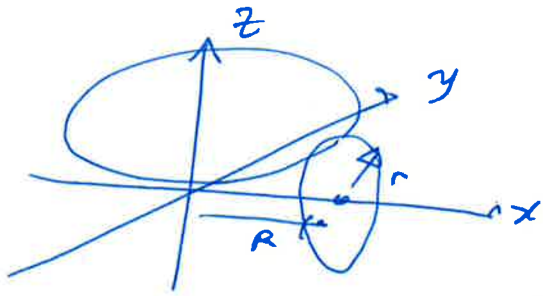
Note that we have an implicit rep^n given by

$$F(x,y) = y - f(x).$$

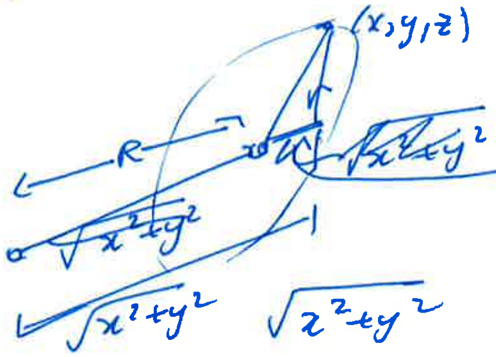
& a parametric by

$$\phi(x) = (x, f(x))$$

The torus



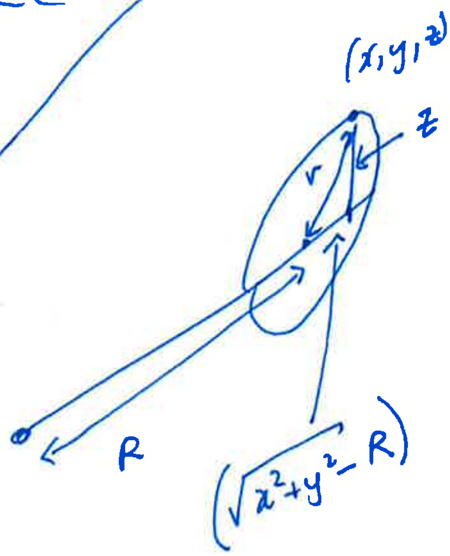
revolve around z to get a torus.



$$\left(\sqrt{x^2+y^2}-R\right)^2+z^2=r^2$$

$$F(x,y,z)=\left((x^2+y^2)^{\frac{1}{2}}-R\right)^2+z^2-r^2$$

Smooth on $U=\mathbb{R}^3-\{z\text{-axis}\}\subseteq\mathbb{R}^3$ open
$(x,y)=(0,0)$



$$\frac{\partial F}{\partial x}=\frac{2\left((x^2+y^2)^{\frac{1}{2}}-R\right)\cancel{\tfrac{1}{2}}\,\cancel{2}x}{(x^2+y^2)^{\frac{1}{2}}}$$

$$\frac{\partial F}{\partial y}=\Big(\qquad\qquad\Big)y$$

$$\frac{\partial F}{\partial z}=2z$$

Assume $\begin{cases}F'(x,y,z)=0\\ F(x,y,z)=0\end{cases}$ then $z=0$

$\therefore$

$\therefore\quad (x^2+y^2)^{\frac{1}{2}}-R=\pm r\neq 0$

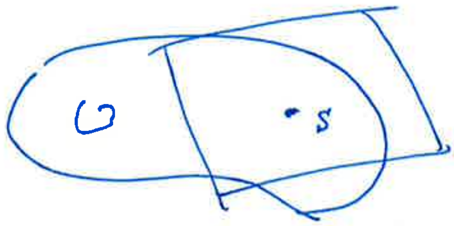
$\therefore\ \dfrac{\partial F}{\partial x}=0=\dfrac{\partial F}{\partial y}\Rightarrow\quad x=y=0.\ \times$

Local parametrisation

$$(\theta, \phi) \rightarrow (\cos\theta(R + r\cos\phi),\ \sin\theta(R + r\cos\phi),\ r\sin\phi)$$

3.1 Tangent space to a submanifold

If S is a submanifold an important object associated to every $s \in S$ is the tangent space at s.



We define it as follows.

Def^n 3.6 Let $s \in S \subseteq \mathbb{R}^N$ be a point in a submanifold. Let $\varepsilon > 0$. Then a smooth $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^N$ is called a <u>smooth path in S through s</u> if

$\gamma(-\varepsilon, \varepsilon) \subseteq S$ &

$\gamma(0) = s$.

~~Previously we denoted by $\gamma'(0)$ the linear map $\gamma'(0): \mathbb{R} \rightarrow \mathbb{R}^N$. We shall abuse notation and also denote the~~