Lecture 13

(3) \Rightarrow (4)

For simplicity assume there is no need to permute indices.

Let \( V \) open \( \in \mathbb{R}^n \)

\( s \in V \)

\( \mathcal{S} \cap \bar{V} = \{ (x, g(x)) \mid x \in U \} \quad \text{g} : U \rightarrow \mathbb{R}^{n-1} \)

Let \( f(x) = (x, g(x)) \quad f : U \rightarrow \mathbb{R}^n \)

\( f \) is 1-1, \( f'(x) = (I, g'(x)) \) is 1-1

\& \( f(U) = \{ (x, g(x)) \mid x \in U \} = \mathcal{S} \cap \bar{V} \).

(4) \Rightarrow (1)

Let \( s \in \mathcal{S} \), \( V \) open in \( \mathbb{R}^n \)

containing \( s \).

\( f : U \rightarrow \mathbb{R}^n \), \( f \) 1-1 \( f' \) 1-1

\( f(U) = \mathcal{S} \cap \bar{V} \)

Choose \( x_0 \in U \) s.t. \( f(x_0) = s \).

\( \uparrow f \)

\( x_0 \in U \)
\( \text{imf}'(x_0) \) is n-dim \( \text{la}\) \( f'(x_0) \equiv 1-1\)

\[ W = (\text{imf}'(x_0))^\perp \text{ is } N-n \text{ dim } \]

Pick a basis \( w^2, \ldots, w^{N-n} \).

Define \( F : U \times \mathbb{R}^{N-n} \rightarrow \mathbb{R}^N \)

\[ F(y, y) = f(x) + \sum_{i=1}^{N-n} w_i y_i \]

\[ F(x_0, 0) = f(x_0) = \lambda \]

Let \( (\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^{N-n} \)

\[ F'(x_0, 0)(\alpha, \beta) = f'(x_0)(\alpha) + \sum_{i=1}^{N-n} w_i \beta_i \]

\[ \text{imf}'(x_0) \subseteq \text{ imf}'(x_0)^\perp \]

\[ \therefore F'(x_0, 0)(\alpha, \beta) = 0 \iff f'(x_0)(\alpha) = 0 \iff \alpha = 0 \]

\[ \sum_{i=1}^{N-n} w_i \beta_i = 0 \iff \beta = 0 \]

\[ \therefore F'(x_0, 0) \text{ is } 1-1. \text{ Also square so } F'(x_0, 0) \text{ is invertible so locally a diffeo} \]

by \( \text{IFT Th} \).

So \( \exists V \subseteq \text{open } U \times \mathbb{R}^{N-n} \) s.t. \( (x_0, 0) \in V \)

s.t. \( F : \tilde{V} \rightarrow F(V) \subseteq \text{open } \mathbb{R}^N \) is a diffeo.
Let \( \bar{V} = F^{-1}(V) \cap \tilde{V} \). Open.

Then \( F : \bar{V} \rightarrow F(\bar{V}) \subseteq \mathbb{R}^n \) is also a cut-off.

Let \( \phi : F(\bar{V}) \rightarrow \bar{V} \) be the inverse of \( F \).

Let \( z \in F(\bar{V}) \cap S \) then

\[ z = f(x) = F(x,0) \]

\[ \phi(z) = (x,0) \]

Also let \( z \in F(\bar{V}) \) & \( \phi(z) = (x,0) \)

Then \( z = F(x,0) = f(x) \), \( z \in S \).

\[ S \cap F(\bar{V}) = \{ z \mid \phi^{k+1}(z) = \ldots = \phi^n(z) \} \]
Hence \( F'(x_0,0) \) is invertible by
\
\[ \text{det} F'(x_0,0) \]
\
Let \( \phi \) be a local inverse for \( F \). If \( (u, z) \in S \) then
\
\[ z = F(x,0) \]
\
If \( (x, y) \in S \), \( y = F(x,0) \)
\
\[ \phi(x, y) = \phi(x, F(x,0)) = (x, 0) \]
\
\[ \phi^{-1}(x, 0) = \ldots = \phi^{-N}(x, 0) = 0 \]
\
\[ \phi^{-N}(x, 0) = \ldots = 0 \quad \phi(u) = (x, 0) \quad u = F(x, 0) \in S. \]

Corollary 3.3

If \( U \subseteq \mathbb{R}^n \) and \( F : U \rightarrow \mathbb{R}^{n+m} \) is smooth with \( F(x) \) onto \( \forall x \in S = F^{-1}(0) \) then \( S \) is a submanifold of dimension \( n \).

Defn 3.4 Let \( S \subseteq \mathbb{R}^n \) be a submanifold.

If \( U \subseteq \mathbb{R}^n \), \( U \) open, and \( F : U \rightarrow \mathbb{R}^{n+m} \) smooth \( F(S) \) onto \( \forall S \subseteq U \) \( S \cap U = \emptyset \)

then \( F \) is called a local defining eqs. for \( S \)

Defn 3.5 Let \( S \subseteq \mathbb{R}^n \) be a submanifold. If \( U \subseteq \mathbb{R}^n \) open, \( V \subseteq \mathbb{R}^n \) open and \( f : U \rightarrow V \) is smooth, 1-1, \( f'(x) \) is 1-1 \( \forall x \in U \) \& \( f(U) = V \) \& \( S \) then \( f \) is called a local parameterization of \( S \)

Note: Neither of these are unique

\( \circ \) If \( N=3, n=2 \) \( F : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)

\( F'(0) \) is a submanifold \( \iff f'(x) \neq 0 \ \forall x \in F^{-1}(0) \)
Example $f : U \to \mathbb{R}^{n-m}$ smooth

then $\text{graph}(f) = \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{R}^{n-m}$

is a submanifold of dimension $n$.

This follows from Theorem 3.2.

Note that we have an implicit representation given by

$$ F(x, y) = y - f(x). $$

& a parametric representation

$$ \phi(x) = (x, f(x)). $$
The task is to revolve around \( z = 0 \) to get a torus.

\[
(x^2 + y^2)^{\frac{1}{2}} - R = z^2 - r^2
\]

Smooth on \( U = \mathbb{R}^3 - \{ z \text{-axis} \} \subset \mathbb{R}^3 \)

\((x, y) = (0, 0)\) open

\[
\frac{\partial F}{\partial x} = 2 \left( (x^2 + y^2)^{\frac{1}{2}} - R \right) \frac{1}{2} \cdot \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}
\]

\[
\frac{\partial F}{\partial y} = \left( \frac{1}{(x^2 + y^2)^{\frac{1}{2}}} \right) y
\]

\[
\frac{\partial F}{\partial z} = 2z
\]

Analyze \( \int F(x, y, z) = 0 \) then \( z = 0 \)

\[
F(x, y, z) = 0
\]

\[
(x^2 + y^2)^{\frac{1}{2}} - R = \pm r \neq 0
\]

\[
\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} \Rightarrow x = y = z = 0.
\]
Local parametrization

\[(\theta, \phi) \rightarrow (\cos \theta (r + r \cos \phi), \sin \theta (r + r \cos \phi), r \sin \phi)\]

3.1 Tangent space to a submanifold

If \(S\) is a submanifold, an important object associated to every \(s \in S\) is the tangent space at \(s\).

We define it as follows.

**Def 3.6** Let \(s \in S\) be a point on a submanifold. Let \(\epsilon > 0\). Then a smooth \(\gamma: (-\epsilon, \epsilon) \rightarrow \mathbb{R}^n\) is called a smooth path on \(S\) through \(s\) if

\[\gamma(0) = s\]

Previously, we denoted by \(\gamma'(0)\) the linear map \(\gamma'(0): \mathbb{R} \rightarrow \mathbb{R}^n\) we call the above notation and also denote the