Lecture 12

12/1

IMPROVEMENT TO IMPLICIT FUNCTION THM

$\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^{n+m}$ $(x_0, y_0) \in \mathcal{U}$ $F : \mathcal{U} \to \mathbb{R}^m$ C^k

$F(x_0, y_0) = 0$ $\frac{\partial F}{\partial y}(x_0, y_0)$ invertible.

$\exists V \subseteq \mathbb{R}^n$ ~~s.t.~~ & $f : V \to \mathbb{R}^m$ s.t.

$F(x, f(x)) = 0.$

Need something slightly stronger.

$\exists$ $\tilde{V} \ni (x_0, y_0)$ open open $f : V \to \mathbb{R}^m$
& V open in $\mathbb{R}^n$

s.t. $\tilde{V} \cap \{(x,y) \mid F(x,y) = 0\}$

$= \{(x, f(x)) \mid x \in V\}$

We actually prove most of this.

We had $\tilde{V}$ & V. & $F(x, f(x)) = 0$

$\therefore$ $\{(x, f(x)) / x \in V\} \subseteq \tilde{V} \cap \{(x,y) / F(x,y) = 0\}$

For the other way: If for $(x,y) \in \tilde{V}$ then $F(x,y) = 0$ & then

$G(x,y) = (x, 0)$. $\therefore$ $G^{-1}(x, 0) = (x, y)$

$\quad\quad //$

$(x, g(x,0)) = (x, y)$

$\therefore y = f(x).$ //

Example from Implicit Fn Th^n

(Before submanifold)

$\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\qquad F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$

$F(x,y) = y - \phi(x)$ $\qquad$ U anywhere on graph

$\frac{\partial F}{\partial y} = I_{\mathbb{R}^m}$ invertible

$G(x,y) = (x, \overset{F(x,y))}{\cancel{y - \phi(x))}}) = (x, y - \phi(x))$

$G^{-1}(u,v) = (u, v + \phi(u))$

Check this

$g(x,y) = y + \phi(x)$

$f(x) = g(x,0) = \phi(x).$ //

Basic facts about linear algebra

$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ ~~$L(x) =$~~ acting on columns

$\ker L = \{ v \in \mathbb{R}^n \mid Lv = 0 \}$ subspace of $\mathbb{R}^n$

$=$ nullspace of L

$\operatorname{im} L = \{ L(v) \mid v \in \mathbb{R}^n \}$ subspace of $\mathbb{R}^m$

$=$ column space of L = span of columns of L.

$\boxed{\dim \ker L + n = \dim \operatorname{im} L}$

~~If $L : \mathbb{R}^n \to \mathbb{R}^n$ then~~

~~L invertible $\iff \dim \ker L = 0 \iff L$ 1-1~~

~~$\iff \dim \operatorname{im} L = n \iff L$ onto~~

$\dim \ker L = 0 \iff L$ 1-1

$\dim$ ~~ker~~ $\operatorname{Im} L = m \iff L$ onto

If $m = n$ $L : \mathbb{R}^n \to \mathbb{R}^n$

L invertible $\iff \dim \ker L = 0$

$\iff \dim \operatorname{Im} L = n$

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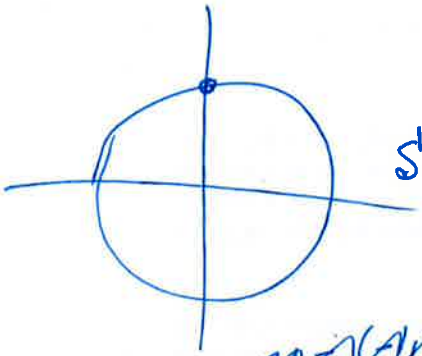
~~E~~ Example for Th^m 3.2



S — ~~A~~ a submanifold

~~$\mathcal{U} = \ldots \cap \ldots$ S get~~

~~$\mathcal{U} = (-1,1)\times(0,\infty)$ ... $\phi(x,y) = (x, y - \sqrt{1-x^2})$~~

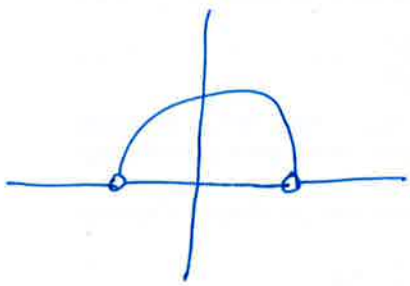
~~$\phi : \mathcal{U} \to$~~

~~Imp~~ Implicit $\quad F : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$F(x,y) = x^2 + y^2 - 1$$

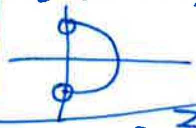
Graphical $\quad f : (-1,1) \longrightarrow \mathbb{R}$

$$f(x) = \sqrt{1-x^2}$$



graph of f.

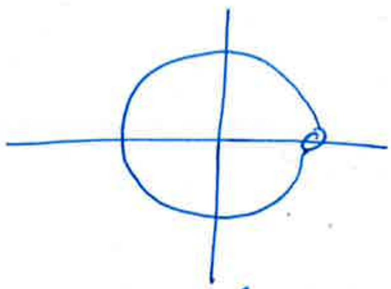
Notice not all of it $(x, f(x))$. Same is ~~$(f(y), y$~~ $(g(y), y)$



Parametric $\quad$ ~~...~~ $f : [0, 2\pi) \longrightarrow \mathbb{R}^2$

$$f(t) = (\cos(t), \sin(t))$$

image(f) =



Can't always get parameter ~~for whole~~ with image all of S. Definitely not in this case

Recall

$S \subseteq \mathbb{R}^N$ is a submanifold if

$\forall s \in S \ \exists\, \mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^N$ $\mathcal{U} \ni s$ & $\phi : \mathcal{U} \to \mathbb{R}^N$

$\phi(\mathcal{U})$ open ϕ a diffeo^n : $\mathcal{U} \to \phi(\mathcal{U})$

$$S \cap \mathcal{U} = \{ x \mid (\phi^{n+1}(x), \ldots, \phi^N(x)) = 0 \}$$

Theorem 3.2

Let $S \subseteq \mathbb{R}^N$. The following are equivalent

① S is a smooth submanifold of dim^n n.

② $\forall s \in S$ $\exists\, \mathcal{U}$ open in $\mathbb{R}^N$, $\mathcal{U} \ni s$ &

$F : \mathcal{U} \to \mathbb{R}^{N-n}$ smooth such that $F'(x)$

is onto $\forall x \in S$ &

$$S \cap \mathcal{U} = \{ x \in \mathcal{U} \mid F(x) = F(s) \}$$

"level set of F"

$\boxed{\text{IMPLICIT}}$

③ $\forall s \in S$ $\exists\, V \subseteq \mathbb{R}^N$ open with $s \in V$ &

a choice of indices $j_1, \dots, j_n \in \{1, \dots, N\}$

~~[illegible]~~ such that $i_1 \cdots i_{N-n}$

are the remaining indices & we write

$$z = (\underbrace{z^{j_1} \cdots z^{j_n}}_{x}, \underbrace{z^{i_1} \cdots z^{i_{N-n}}}_{y})$$

then $S \cap V = \{ (x, f(x)) \mid x \in \mathcal{U} \}$

for $f : \underset{\substack{\cap \text{open} \\ \mathbb{R}^n}}{\mathcal{U}} \to \mathbb{R}^{N-n}$ a smooth function.

"locally S is the graph of a function"

$\boxed{\text{GRAPHICAL}}$

(4) $\forall s \in S$ $\exists V \underset{\text{open}}{\subseteq} \mathbb{R}^N$ s.t. $s \in V$, $\mathcal{U} \subseteq \mathbb{R}^n$ open & ~~sm~~ $\exists$ smooth $f: \mathcal{U} \to \mathbb{R}^N$ which is 1-1 & s.t. $f'(x)$ is 1-1 $\forall x \in \mathcal{U}$ & $f(\mathcal{U}) = S \cap V$.

$\boxed{\text{PARAMETRIC}}$

$\boxed{\text{DO EXAMPLES BEFORE PROOF}}$

Proof

(1) $\Rightarrow$ (2) Let $s \in S$. By defn we have $\mathcal{U} \subseteq \mathbb{R}^N$, $\mathcal{U}$ open, $s \in \mathcal{U}$ & $\phi: \mathcal{U} \to \mathbb{R}^N$ s.t.

$$S \cap \mathcal{U} = \{x \in \mathcal{U} \mid (\phi^{n+1}(x) \ldots \phi^N(x)) = 0\}$$

Let $F(x) = (\phi^{n+1}(x), \ldots, \phi^N(x)) \in \mathbb{R}^{N-n}$

$$F: \mathcal{U} \to \mathbb{R}^{N-n}.$$

$$S \cap \mathcal{U} = \{x \in \mathcal{U} \mid F(x) = F(s) = 0\}$$

Why is $F'(x)$ onto?

~~Write~~ $\phi(x) = (\phi^1 \ldots \phi^n, F) = (\psi, F)$

$$\phi'(x)(h) = (\psi'(x)(h), F'(x)(h))$$

ϕ a diffeo^{m} $\Rightarrow$ $\phi'(x)$ invertible

$\Rightarrow$ $\phi'(x)$ onto

$\Rightarrow$ F' onto.

(2) $\Rightarrow$ (3)

Fix $s \in S$ we have

$$\underbrace{\{ F'(s)(e^1), \dots, F'(s)(e^N) \}}_{\text{columns!}} \text{ span } \mathbb{R}^{N-n}$$

So $\exists\ i_1 \dots i_{N-n}$ s.t.

$$F'(s)(e^{i_1}) \dots F'(s)(e^{i_{N-n}}) \text{ is a basis of } \mathbb{R}^{N-n}$$

Let $j_1 \dots j_n$ be the other indices

$$\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^{N-n}$$

$$z = (\underbrace{z^{j_1} \dots z^{j_n}}_{x}, \underbrace{z^{i_1} \dots z^{i_{N-n}}}_{y})$$

Write $\tilde{F}(x,y) = F(z)$ then

$$\frac{\partial F}{\partial y}(s)(v^1, \dots, v^{N-n}) = \sum \frac{\partial \tilde{F}}{\partial z^{i_j}}(s)\, v^j$$

$$= \sum \underbrace{\tilde{F}'(s)(e^{i_j})}_{\text{linearly ind}} v^j$$

$$\therefore \frac{\partial F}{\partial y}(s)(v^1 \dots v^{N-n}) = 0 \iff (v^1, \dots, v^{N-n}) = 0$$

$$\therefore \ker \frac{\partial F}{\partial y}(s) = 0 \quad \therefore \frac{\partial F}{\partial y}(s) \text{ invertible}$$

Now use Implicit fn Th^m