Lecture 11

Recall IFThm $\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n$

$f : \mathcal{U} \to \mathbb{R}^n$ C^k $k \geq 1$ $a \in \mathcal{U}$ s.t.

$f'(a)$ invertible. Then $\exists V \subseteq \mathbb{R}^n$ open $a \in V \subseteq \mathcal{U}$.

$f(V) \subseteq_{\text{open}} \mathbb{R}^n$. $f : V \to f(V)$ invertible.

$f^{-1} : f(V) \to f$ C^k & $(f^{-1})'(f(a)) = [f'(a)]^{-1}$

~~Recall~~ $f : \mathbb{R}^n \to \mathbb{R}^m$ cts $\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^m \Rightarrow f^{-1}(\mathcal{U}) \underset{\text{open}}{\subseteq} \mathbb{R}^n$.

Corollary 2.22 (Open Mapping Thm)

Let $\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n$ & $f : \mathcal{U} \to \mathbb{R}^n$ ~~differentiable~~ C^k $k \geq 1$

with $f'(x_0)$ invertible $\forall x \in \mathcal{U}$. Then

$f(\mathcal{U}) \underset{\text{open}}{\subseteq} \mathbb{R}^n$.

Proof ~~Let $x \in \mathcal{U}$ the $\exists V \underset{\text{open}}{\subseteq} \mathbb{R}^n$ s.t.~~

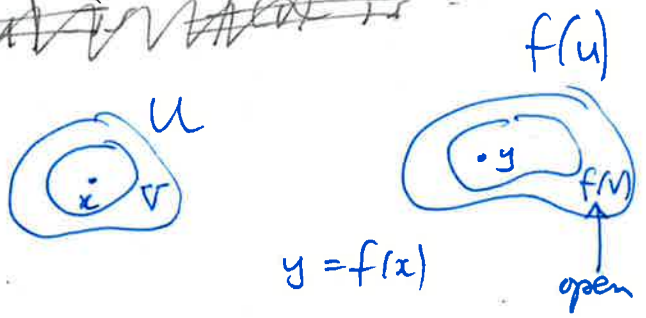
~~$f(V) \underset{\text{open}}{\subseteq} \mathbb{R}^n$ & $x \in V$ $\therefore$~~ $f(\mathcal{U})$ $f(x)$.

Let $y = f(x) \in f(\mathcal{U})$. Then $\exists V$ open in $\mathbb{R}^n$

s.t. $x \in V \subseteq \mathcal{U}$ & $f(x) \in f(V) \subseteq f(\mathcal{U})$

$f(V) \underset{\text{open}}{\subseteq} \mathbb{R}^n$ $\therefore$ ~~$f(\mathcal{U})$ is $f(x)$~~ is

$f(\mathcal{U})$ is open //



* Remember $f : \mathbb{R}^n \to \mathbb{R}^m$ cts. y

$\Rightarrow \left(W \underset{\text{open}}{\subseteq} \mathbb{R}^m \Rightarrow f^{-1}(W) \underset{\text{open}}{\subseteq} \mathbb{R}^n\right)$

The Implicit Function Th^m generalises this.

<u>Th^m 2.24 (Implicit Function Th^m)</u>

Let U be open in $\mathbb{R}^{n+m}$ & $F : U \to \mathbb{R}^m$ be C^k. If $F(x_0, y_0) = 0$ &

$$\frac{\partial F}{\partial y}(x_0, y_0) = \begin{pmatrix} \frac{\partial F^1}{\partial y_1}(x_0,y_0) & \cdots & \frac{\partial F^1}{\partial y_m}(x_0,y_0) \\ \vdots & & \\ \frac{\partial F^m}{\partial y_m}(x_0,y_0) & \cdots & \frac{\partial F^m}{\partial y_m}(x_0,y_0) \end{pmatrix}$$

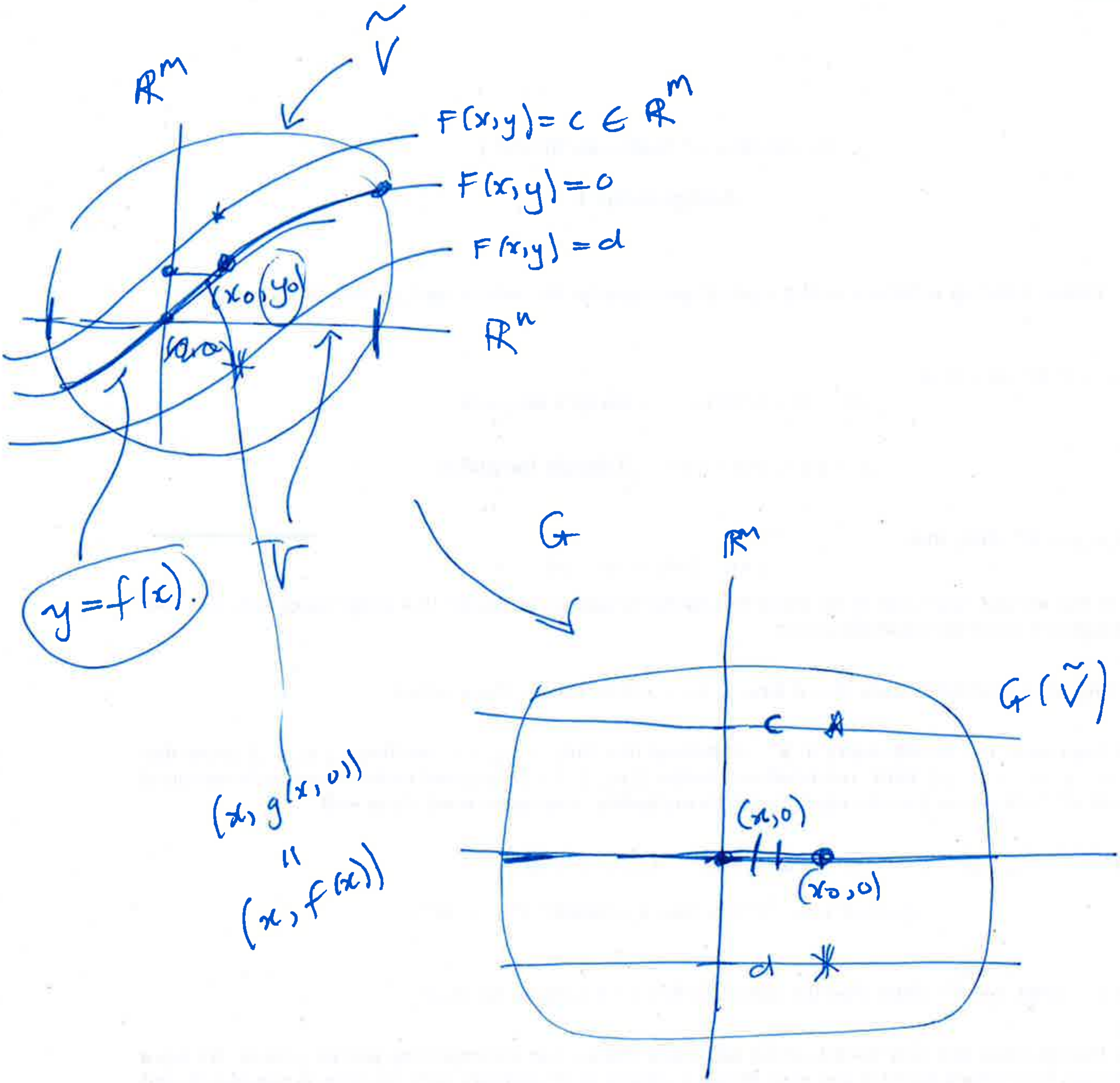
is <u>not singular</u> then $\exists\ V \subseteq \mathbb{R}^n$ open containing x_0 & a C^k function $f : V \to \mathbb{R}^m$ such that $F(x, f(x)) = 0 \quad \forall x \in V$.

<u>Proof</u> (Use <u>IFTh^m</u>)

Define $G : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{n+m}$

$$G(x,y) = (x,\ F(x,y)) \qquad (- \ C^k.)$$

$$G'(x,y) = \left[\begin{array}{c|c} I & \frac{\partial x^i}{\partial y^j} = 0 \\ \hline \frac{\partial F^i}{\partial x^j} & \frac{\partial F}{\partial y} \end{array}\right]$$



$\tilde{V}$
$\mathbb{R}^m$
$F(x,y) = c \in \mathbb{R}^m$
$F(x,y) = 0$
$F(x,y) = d$
(x_0, y_0)
$\mathbb{R}^n$
$y = f(x)$.
V
$(x, g(x,0))$
$\shortparallel$
$(x, f(x))$
G
$\mathbb{R}^m$
$G(\tilde{V})$
c
$(x,0)$
$(x_0,0)$
d

Def^n 2.23 If $U, V \underset{\text{open}}{\subseteq} \mathbb{R}^n$ & $f : U \to V$ is C^k & with C^k inverse we call f a C^k <u>diffeomorphism</u> (diffeo^m)

<u>Compare</u> To ~~f2~~: X, Y topological spaces $f : X \to Y$ cts, invertible, f^{-1} cts is a homeo^m. ok

For the ~~...~~ Implicit Function Th^m we need some notation. ~~If~~ $x = (x^1, \dots x^n) \in \mathbb{R}^n$ & $y = (y^1, \dots, y^m) \in \mathbb{R}^m$ we let

$$(x, y) = (x^1, \dots, x^n, y^1, \dots, y^m) \in \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$$

Notice that if $f : \mathbb{R}^n \to \mathbb{R}^m$ $\text{graph}(f) = \{(x, f(x)) \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^{n+m}$

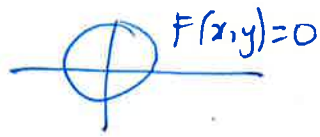
Recall the classical result that if

$$F : \mathbb{R}^2 \to \mathbb{R} \quad F(0,0) = 0 \;\; \& \;\; \left.\frac{\partial F}{\partial y}\right|_{(0,0)} \neq 0$$

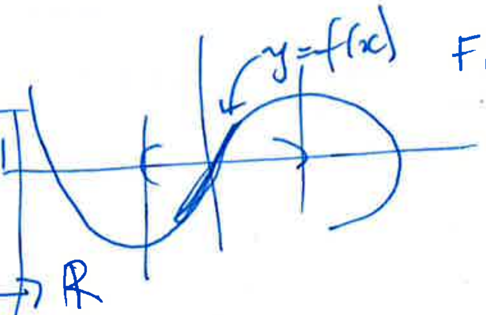
~~F(0,0~~ ~~for F(0,0) = 0~~

~~NB~~

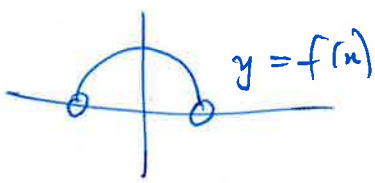
we can locally write $F(x,y) = 0$ as the graph of a function f ie solve



$$F(x, f(x)) = 0$$



$F(x,y) = 0$



EG $F(x,y) = x^2 + y^2 - 1$

$f(x) = \sqrt{1 - x^2}$ $\quad f : (-1,1) \to \mathbb{R}$

So $\det(G(x_0, y_0)) = \det(I) \det\left(\frac{\partial F}{\partial y}(x_0, y_0)\right)$

$\neq 0$.

Hence $\exists\ \underset{\text{open}}{\tilde{V}} \subseteq \mathbb{R}^{n+m}$, $(x_0, y_0) \in \tilde{V}$

$G(\tilde{V})$ open in $\mathbb{R}^{n+m}$ &

$G: \tilde{V} \to G(\tilde{V})$ is invertible &

G^{-1} is C^k. DRAW PICTURE

If $(u, v) \in G(\tilde{V})$

then $(u, v) = (x, F(x, y))$ for some $(x, y) \in \tilde{V}$

& $G^{-1}(u, v) = (x, y)$.

$\therefore G^{-1}(u, v) = (u, g(u, v))$. $g: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ C^k

Let $V = \{ x \in \mathbb{R}^n \mid (x, 0) \in \tilde{V} \}$

$\boxed{\text{(Ex } \tilde{V} \text{ open} \Rightarrow V \subseteq \mathbb{R}^n \text{ is open.)}}$

Define $f: V \to \mathbb{R}^m$ by

$f(x) = g(x, 0)$ $\therefore f$ C^k &

~~$(x, 0) \longmapsto (x, F(x, 0))$~~

$G G^{-1}(x, 0) = (x, 0)$ $\quad G^{-1}(x, 0) = (x, \underset{g(x,0)}{f(x)}) = (x, f(x))$

$G G^{-1}(x, 0) = (x, F(x, f(x)))$

$\overset{\|}{(x, 0)}$ $\quad \therefore F(x, f(x)) = 0$

§3 SUBMANIFOLDS

Def^n 3.1 A subset $S \subseteq \mathbb{R}^N$ is called a *submanifold of dimension n* ~~of~~ if $\exists\, n \geq 0$ s.t. $\forall s \in S$ $\exists\, \mathcal{U} \subseteq_{open} \mathbb{R}^N$ & $\phi : \mathcal{U} \to \mathbb{R}^N$ s.t. $\phi(\mathcal{U}) \subseteq_{open} \mathbb{R}^N$, $\phi : \mathcal{U} \to \phi(\mathcal{U})$ a diffeo^m

$$\& \quad S \cap \mathcal{U} = \{ x \in \mathcal{U} \mid \phi^{(n+1)}(x) = \phi^{(n+2)}(x) = \dots = \phi^N(x) = 0 \}$$

~~We will~~ If necessary we ~~require~~ specify ϕ to be C^k $(k = 1, 2, \dots \infty)$ and call S a C^k submanifold but usually assume S is C^∞ or smooth

IDEA: