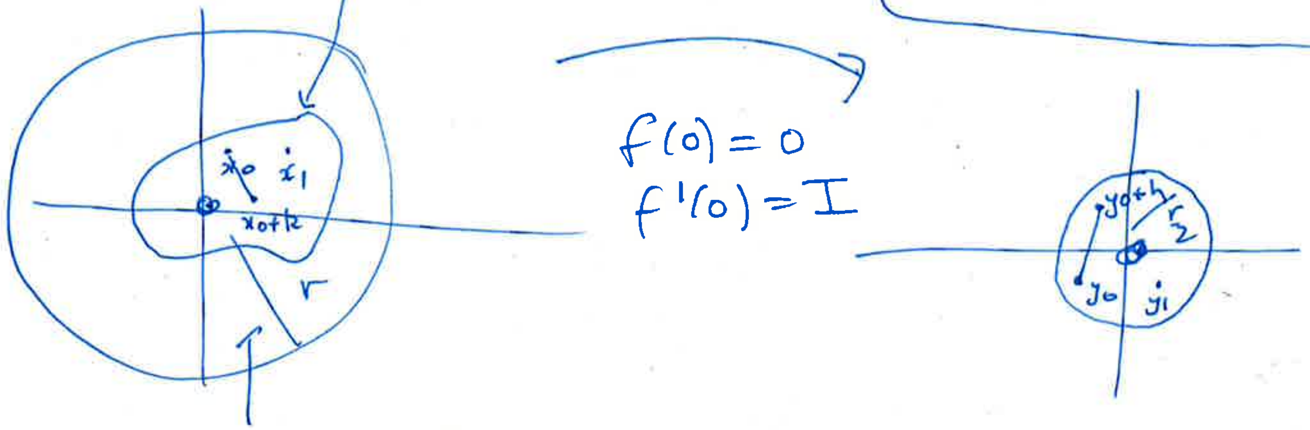
Lecture 10

Recall

Notes from Inv F^n Thm beginning

$V = B(0,r) \cap f^{-1}(B(0, r/2))$

f



$B(0,r)$

$f(0) = 0$

$f'(0) = I$

$\|I - f'(x)\|_2 < \frac{1}{2}$

& Corollary If $h : \mathcal{U} \to \mathbb{R}$ is diff'ble & $\|h'(\xi)\| \leq \varepsilon \ \forall \xi \in [x_0, x_1]$ then $\|h(x_0) - h(x_1)\| < \varepsilon \|x_0 - x_1\|$

Claim If $x_0, x_1 \in \overline{B}(0,r)$ then

$$\boxed{\|(x_1 - x_0) - (f(x_1) - f(x_0))\| < \frac{1}{2}\|x_1 - x_0\|}$$

Proof ~~the~~ Let $h(x) = x - f(x)$ & use Corollary to MVT as

$$h'(x)(h) = (I - f'(x))(h) \quad //$$

We want to show that if $\|y\| < r/2$ then $\exists!\, x \in B(0,r)$ s.t. $f(x) = y$

Let $g(x) = x + y - f(x)$ then

$$g(x) = x \iff f(x) = y$$

Want to use CMTh : $g : \overline{B}(0,r) \to \overline{B}(0,r)$

(i) $\underline{g(\overline{B(0,r)}) \subseteq \overline{B(0,r)}}$

$$\|g(x)\| = \|x - f(x) + y\|$$
$$\leq \|x - f(x)\| + \|y\|$$
$$< \frac{1}{2}\|x\| + \|y\| < r/2 + r/2 = r$$

(ii) $\underline{g \text{ is a contraction}}$

$$\|g(x_0) - g(x_1)\| = \|x_0 + y - f(x_0) - (x_1 + y + f(x_1))\|$$
$$= \|x_0 - x_1 - (f(x_0) - f(x_1))\| < \frac{1}{2}\|x_1 - x_0\|$$

$\therefore$ g is a contraction. ~~$\exists!$ fixed pt~~

$\therefore \exists!$ fixed point by CMTh Prop 1.15

x.

Note $\|x\| = \|g(x)\| < r$.

$\therefore \underline{x \in B(0,r)}$

may have more components

Let $V = B(0,r) \cap f^{-1}(B(0, r/2))$

open!

$f : V \to f(V) = B(0, r/2)$

is 1-1 becau if $y \in f(V)$ then & $\|y\| < r/2$.

$y = f(x)$, $\|x\| < r$ ~~$\|y\| < r/2$~~ $\therefore \exists!\, x$ st $f(x) = y$

is t & $\|x\| < r$ by CMTh

Hence $f^{-1} : f(V) \to V$ is well-defined

(i) f^{-1} is continuous

Let $f(x_0) = y_0$ $\quad f(x_1) = y_1$

$x_0, x_1 \in V = B(0,r) \cap f^{-1}(B(0, r/2))$

$$\|x_0 - x_1\| \leq \|x_0 - x_1 - f(x_0) - f(x_1)\| + \|f(x_0) - f(x_1)\|$$

$$\leq \frac{1}{2}\|x_0 - x_1\| + \|f(x_0) - f(x_1)\|$$

$\therefore$ $\frac{1}{2}\|x_0 - x_1\| \leq \|f(x_0) - f(x_1)\|$

$\therefore$ $\|f^{-1}(y_0) - f^{-1}(y_1)\| \leq 2\|y_0 - y_1\|$

This is enough to make f^{-1} ct an it.

$y_n \to y$ the

$\|f^{-1}(y_n) - f^{-1}(y_1)\| \leq 2\|y_n - y\| \to 0$

$\therefore \searrow 0.$

(ii) f^{-1} is differentiable.

fix $y_0 \in f(V) = B(0, r/2)$ $\quad x_0 = f^{-1}(y_0)$

Let $x_0 + k = f^{-1}(y_0 + h)$. $\quad (y_0 + h \in B(0, r/2))$

We expect $(f^{-1})'(y_0) = [f'(x_0)]^{-1}$

Consider

$$\frac{\| \underbrace{f^{-1}(y_0+h) - f^{-1}(y_0)} - f'(x_0)^{-1}(h)\|}{\|h\|}$$

$$\leq \frac{\| k - f'(x_0)^{-1}(\overbrace{f(x_0+k)-f(x_0)})\|}{\|h\|}$$

$$\leq \frac{\|k\|}{\|h\|}\left(\frac{\|f'(x_0)^{-1}\|_2 \, \|f(x_0+k)-f(x_0)-f'(x_0)k\|}{\|k\|}\right.$$

But $\|k\| = \|x_0 + k - x_0\|$

$= \|f^{-1}(y_0+h) - f^{-1}(y_0)\|$

$\leq 2\|y_0+h-y_0\| = 2\|h\|$

$\therefore \leq \quad 2\|f'(x_0)^{-1}\|_2 \dfrac{\|f(x_0+k)-f(x_0)-f'(x_0)k\|}{\|k\|}$

$\longrightarrow 0$ as $\cancel{\|k\|} \to 0$

$\|h\| \to 0$

$\therefore \|k\| \to 0$

To show f^{-1} is C^k we use a trick as for chain rule

~~$y \longmapsto f^{-1}(y)$~~ $J(f^{-1})(y)$

$\cancel{f'(y)}\ (f^{-1})'(y) = \left(f'(\underbrace{f^{-1}(y)}_{}) \right)^{-1}$

$$y \xmapsto[\underset{C}{\uparrow}]{f^{-1}} f^{-1}(y) \xrightarrow[C^{k-1}]{} f'(f^{-1}(y)) \xrightarrow[\underset{C^\infty}{\uparrow}]{-1} f'(f^{-1}(y))^{-1}$$

C^0

since

As f^{-1} is C^0 we deduce ∂ is $\&$

derivative are C^0 $\therefore$ f^{-1} is C^1

If f^{-1} is C^{k-1} $\rightarrow$ derivatives are C^{k-1}

$\therefore$ f^{-1} is C^k. //

If X is a matrix & $\det(X) \neq 0$

$$(X^{-1})_{ij} = \frac{\text{polynomial in } X_{ij}}{\det X = \text{polynomial in } X_{ij}}$$

$\therefore$ ∞ly differentiable.

Example $\quad f : \mathbb{R} \to \mathbb{R} \quad f(x) = x^3$

$f'(x) = 0.$

f^{-1} exists & is cts but not ~~R~~ differentiable

$f^{-1}(x) = x^{1/3}$

$\not\exists$.

↓ End lecture 10

~~Corollary 2.22 (Open Mapping Thm)~~

~~Let $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$ & $f : U \to \mathbb{R}^n$ differentiable with $f'(x_0)$ invertible $\forall x \in U$. Then $f(U) \underset{\text{open}}{\subseteq} \mathbb{R}^n$.~~

~~Proof~~ ~~Let $x \in U$ the $\exists V \underset{\text{open}}{\subseteq} \mathbb{R}^n$ s.t.~~

~~$f(V) \underset{\text{op}}{\subseteq} f(U)$ ~~$\mathbb{R}^n$~~ & $x \in V$ $\therefore$ $f(x)$.~~

~~Let $y = f(x) \in f(U)$. Then $\exists V$ open in $\mathbb{R}^n$ s.t. $x \in V \subseteq U$ & $f(x) \in f(V) \subseteq f(U)$~~

~~$f(V) \underset{\text{ope}}{\subseteq} \mathbb{R}^n$ $\therefore$ ~~$f(x) \in V$ $f(x)$ is~~~~

~~$f(U)$ is open //~~

* Remember $\quad f : \mathbb{R}^n \to \mathbb{R}^m$ cts.

$\Rightarrow \left(W \underset{\text{open}}{\subseteq} \mathbb{R}^m \Rightarrow f^{-1}(W) \underset{\text{open}}{\subseteq} \mathbb{R}^n \right)$