Solutions to Ass 6 — 2011

1. **Linear independence:** Assume $\sum_{i=1}^{n} \lambda_i \alpha_i = 0$

Then $0 = \left(\sum_{i=1}^{n} \lambda_i \alpha_i\right)(v_j) = \lambda_j$. $\therefore$ linearly independent.

Span: Let $\xi \in V^*$ and consider $\sum_{i=1}^{n} \xi(v_i)\alpha_i$.

$\left(\sum_{j=1}^{n} \xi(v_i)\alpha_i\right)(v_j) = \xi(v_j)$. True for all basis so $\xi = \sum_{j=1}^{n} \xi(v_i)\alpha_i$.

2. (a) From Q4 a: $vol = \left\| \frac{\partial X}{\partial \theta} \times \frac{\partial X}{\partial \phi} \right\| d\hat{x}^1 \wedge d\hat{x}^2$

From soln to Ass 5: $\left\| \frac{\partial X}{\partial \theta} \times \frac{\partial X}{\partial \phi} \right\| = r(R + r\cos\theta)$

(b) $\int vol = \int_0^{2\pi} \int_0^{2\pi} r(R + r\cos\theta)\, d\theta\, d\phi = 4\pi^2 rR$

(c) $R\, vol = \frac{\cos(\phi)}{r(R + r\cos\theta)}\, r(R + r\cos\theta)\, d\hat{x}^1 \wedge d\hat{x}^2$.

$\int R\, vol = \int_0^{2\pi} \int_0^{2\pi} \cos\phi\, d\theta\, d\phi = 0$.

3. (a) $(\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v)$

$= -(\cancel{\alpha(w)\beta(v) - \alpha(v)\beta(w)})$

$= -(\beta(v)\alpha(w) - \beta(w)\alpha(v))$

$= -\beta \wedge \alpha(v, w)$

$\therefore \alpha \wedge \beta = -\beta \wedge \alpha$. $\quad \alpha \wedge \alpha = -\alpha \wedge \alpha \quad \therefore \alpha \wedge \alpha = 0$

(2)

(b) $df = \sum_i df\left(\frac{\partial \psi}{\partial x^i}\right) d\hat{\psi}^i$ as they are dual bases. So $df = \sum \frac{\partial f \circ \psi}{\partial x^i} d\hat{\psi}^i$ from Ass 5 Q1. ($df = f'$).

(c) From lectures: if $\psi(x) = s$

$$d(f\alpha)(s) = \left(\frac{\partial f\alpha_2 \circ \psi}{\partial x^1}(x) - \frac{\partial f\alpha_1 \circ \psi}{\partial x^2}(x)\right) d\hat{\psi}^1_{(s)} \wedge d\hat{\psi}^2_{(s)}.$$

~~Prop~~ x & S

$$= \begin{pmatrix} \frac{\partial f \circ \psi}{\partial x^1}(x)\, \alpha_2 \circ \psi(x) + f \circ \psi_{(x)} \frac{\partial \alpha_2 \circ \psi}{\partial x^2}(x) \\ - \frac{\partial f \circ \psi}{\partial x^2}(x)\, \alpha_1 \circ \psi(x) + f \circ \psi_{(x)} \frac{\partial \alpha_1 \circ \psi}{\partial x^2}(x) \end{pmatrix} d\hat{\psi}^1 \wedge d\hat{\psi}^2$$

$$= \left(\frac{\partial f \circ \psi}{\partial x^1}(x)\, \alpha_2(s) - \frac{\partial f \circ \psi}{\partial x^2}(x)\, \alpha_1(s)\right) d\hat{\psi}^1 \wedge d\hat{\psi}^2$$

$$+ f(s)\, d\alpha$$

But $df \wedge \alpha = \left(\frac{\partial f \circ \psi}{\partial x^1} d\hat{\psi}^1 + \frac{\partial f \circ \psi}{\partial x^2} d\hat{\psi}^2\right)\left(\alpha_1 d\hat{\psi}^1 + \alpha_2 d\hat{\psi}^2\right)$

$$= \left(\frac{\partial f \circ \psi}{\partial x^1} \alpha_2 - \frac{\partial f \circ \psi}{\partial x^2} \alpha_1\right) d\hat{\psi}^1 \wedge d\hat{\psi}^2$$

as required.

(d) $df = \frac{\partial f \circ \psi}{\partial x^1} d\hat{\psi}^1 + \frac{\partial f \circ \psi}{\partial x^2} d\hat{\psi}^2$

$$\therefore ddf = \left(\frac{\partial^2 f \circ \psi}{\partial x^1 \partial x^2} - \frac{\partial^2 f \circ \psi}{\partial x^2 \partial x^1}\right) d\hat{\psi}^1 \wedge d\hat{\psi}^2$$

$$= 0.$$

(e) $\frac{\partial \psi}{\partial x^i}$ & $d\hat{\psi}^i$ are dual. $\left[\frac{\partial \psi}{\partial x^1}, \frac{\partial \psi}{\partial x^2}\right] = d\hat{\psi}^1 \wedge d\hat{\psi}^2$

Follows from Prop^n 6.5. Lemma 6.21

4 a $vd\cancel{\ }\; = vd\left(\frac{\partial\psi}{\partial x^1}, \frac{\partial\psi}{\partial x^2}\right) d\hat{\psi}^1 \wedge d\hat{\psi}^2$

from Q3(e).

$$vd\left(\frac{\partial\psi}{\partial x^1}, \frac{\partial\psi}{\partial x^2}\right) = \left\langle \frac{\partial\psi}{\partial x^1} \times \frac{\partial\psi}{\partial x^2}, n \right\rangle$$

$$= \left\langle \frac{\partial\psi}{\partial x^1} \times \frac{\partial\psi}{\partial x^2}, \frac{\frac{\partial\psi}{\partial x^1} \times \frac{\partial\psi}{\partial x^2}}{\left\|\frac{\partial\psi}{\partial x^1} \times \frac{\partial\psi}{\partial x^2}\right\|} \right\rangle$$

$$= \left\|\frac{\partial\psi}{\partial x^1} \times \frac{\partial\psi}{\partial x^2}\right\|$$

b Q3(e) $\Rightarrow$ $Rvd = \underbrace{Rvd\left(\frac{\partial\psi}{\partial x^1}, \frac{\partial\psi}{\partial x^2}\right)} d\hat{\psi}^1 \wedge d\hat{\psi}^2$

$$= \left\langle -n'\left(\frac{\partial\psi}{\partial x^1}\right) \times -n'\left(\frac{\partial\psi}{\partial x^2}\right), n \right\rangle \quad \text{(Corollary 6.18)}$$

$$= \left\langle \frac{\partial n \circ \psi}{\partial x^1} \times \frac{\partial n \circ \psi}{\partial x^2}, n \right\rangle$$

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5 a $\psi(\theta, \phi) = (a\cos\theta\sin\phi, b\sin\theta\sin\phi, c\cos\phi)$

b $\frac{\partial\psi}{\partial\theta} = (-a\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$

$$\frac{\partial\psi}{\partial\phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$$

$$\frac{\partial\psi}{\partial\phi} \times \frac{\partial\psi}{\partial\theta} = (bc\cos\theta\sin\phi, ac\sin\theta\sin\phi, ab\cos\phi)$$

Surface area a ellipsoid

$$\int_0^{\pi}\int_0^{2\pi} \sqrt{b^2c^2\cos^2\theta\sin^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\phi}\; d\theta\, d\phi$$