Geometry of Surfaces 2011

Assignment 5 — Solutions

1*. Notice that if f was defined on all of $\mathbb{R}^3$ then this would just be the chain rule. But as f is not defined on all of $\mathbb{R}^3$ we cannot compute the three partial derivatives of f. Instead we recall first from Corollary 2.12 that we have

$$\frac{\partial f \circ \psi}{\partial x^i}(x) = (f \circ \psi)'(x)(e^i).$$

Then we do have a chain rule for functions defined only on submanifolds which is give in Proposition 3.16. It tells us that

$$(f \circ \psi)'(x)(e^i) = f'(s)\left(\psi'(x)(e^i)\right) = f'(s)\left(\frac{\partial \psi}{\partial x^i}(x)\right)$$

which gives the result.

You could instead extend f to $\tilde{f}$ defined (locally) on an open set in $\mathbb{R}^3$, apply the chain rule and then observe that $\tilde{f}'(s)(v) = f'(s)(v)$ whenever $v \in T_s\Sigma$.

2. (a) $\psi'(t) = (-\sin(t), \cos(t), 0)$ so that $\|\psi'(t)\| = 1$ and hence ψ is an arc-length parametrisation.
(b) We have $T(t) = (-\sin(t), \cos(t), 0)$. Thus $T'(t) = (-\cos(t), -\sin(t), 0)$. As the parametrisation is by arc-length we have and $T'(t)$ has length one we obtain $N(t) = (-\cos(t), -\sin(t), 0)$. $B = T \times N = (0, 0, 1)$. Using the Frenet formula it follows that $\kappa = 1$ and $\tau = 0$.

3*. We have $\gamma(t) = (3t^2, 6t, 6) = 3(t^2, 2t, 2)$ so $\|\gamma'(t)\| = 3(t^4 + 4t^2 + 4)^{1/2} = 3(t^2 + 2)$. Therefore

$$T = \frac{1}{(t^2+2)}(t^2, 2t, 2),$$

so $T' = -(t^2+2)^{-2}(2t)(t^2, 2t, 2) + (t^2+2)^{-1}(2t, 2, 0) = (t^2+2)^{-2}(4t, 4-2t^2, -4t) = 2(t^2+2)^{-2}(2t, 2-t^2, -2t)$, giving

$$N = \frac{1}{(t^2+2)}(2t, 2-t^2, -2t).$$

Note that $\dot{T} = \kappa N$ and $\dot{T} = \|\gamma'\|^{-1} T'$ so

$$\kappa = \frac{2}{3(t^2+2)^2}.$$

Next,

$$\begin{aligned} B &= T \times N \\ &= \frac{1}{(t^2+2)^2}\left[(t^2, 2t, 2) \times (2t, 2-t^2, -2t)\right] \\ &= \frac{1}{(t^2+2)^2}(-2t^2 - 4, 2t^3 + 4t, -t^4 - 2t^2) \\ &= \frac{1}{(t^2+2)}(-2, 2t, -t^2). \end{aligned}$$

Thus $B' = 2(t^2+2)^{-2}(2t, 2-t^2, -2t)$ (since B is the same as T with the first and third entries swapped and multiplied by -1) giving $\dot{B} = (2/3)(t^2+2)^{-3}(2t, 2-t^2, -2t) = (2/3)(t^2+2)^{-2} N$ and hence

$$\tau = \frac{-2}{3(t^2+2)}.$$

4. The detailed calculations are on the web page but we sketch here the answers. Firstly $\gamma'(t) = 3(1-t^2, 2t, 1+t^2)$ and $\|\gamma'(t)\| = 3\sqrt{2}(1+t^2)$. So this is not arc-length parametrised. Then

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{1}{\sqrt{2}(1+t^2)}(1-t^2, 2t, 1+t^2).$$

Next we have $T'(t) = \sqrt{2}(1+t^2)^{-1}(-2t, 1-t^2, 0)$ so that

$$N = \frac{T'(t)}{\|T'(t)\|} = \frac{1}{1+t^2}(-2t, 1-t^2, 0).$$

Also

$$\dot{T} = \frac{T'}{\|\gamma'(t)\|} = \frac{1}{3(1+t^2)^3}(-2t, 1-t)^2, 0) = \frac{1}{3(1+t^2)^2}N.$$

Finally

$$B = T \times N = \frac{-1}{\sqrt{(1+t^2)}}(1-t^2, 2t, -(1+t^2))$$

and

$$\dot{B} = \frac{-1}{3(1+t^2)^2}(-2t, 1-t^2, 0) = -\frac{1}{3(1+t^2)^2}N.$$

So that

$$\kappa = \frac{1}{3(1+t^2)^2} \qquad \text{and} \qquad \tau = \frac{1}{3(1+t^2)^2}.$$

5*.

(a) We have

$$\frac{\partial \chi}{\partial \theta} = (-\sin(\theta)(R + r\cos(\phi)), \cos(\theta)(R + r\cos(\phi)), 0)$$

and

$$\frac{\partial \chi}{\partial \phi} = (-r\cos(\theta)\sin(\phi), -r\sin(\theta)\sin(\phi), r\cos(\phi)).$$

So

$$\frac{\partial \chi}{\partial \theta} \times \frac{\partial \chi}{\partial \phi} = r(R + r\cos(\phi)(\cos(\phi)\cos(\theta), \cos(\phi)\sin(\theta), \sin(\phi)).$$

Hence

$$n = (\cos(\phi)\cos(\theta), \cos(\phi)\sin(\theta), \sin(\phi)).$$

If we evaluate at $(0,0)$ that is the point $(r+R,0,0)$ and we get $(1,0,0)$ which is pointing outwards.

(b) Calculating we have

$$\frac{\partial^2 \chi}{\partial \theta^2} = (-\cos(\theta)(R + r\cos(\phi)), -\sin(\theta)(R + \cos(\phi)), 0)$$
$$\frac{\partial^2 \chi}{\partial \theta \partial \phi} = (r\sin(\theta)\sin(\phi), -r\cos(\theta)\sin(\phi), 0)$$
$$\frac{\partial^2 \chi}{\partial \phi^2} = (-r\cos(\theta)\cos(\phi), -r\sin(\theta)\cos(\phi), -r\sin(\phi)).$$

It follows that

$$\alpha\left(\frac{\partial \chi}{\partial \theta}, \frac{\partial \chi}{\partial \theta}\right) = \left\langle \frac{\partial^2 \chi}{\partial \theta^2}, n \right\rangle = -\cos(\phi)(R + \cos(\phi)).$$
$$\alpha\left(\frac{\partial \chi}{\partial \theta}, \frac{\partial \chi}{\partial \phi}\right) = \left\langle \frac{\partial^2 \chi}{\partial \theta \partial \phi}, n \right\rangle = 0.$$
$$\alpha\left(\frac{\partial \chi}{\partial \phi}, \frac{\partial \chi}{\partial \phi}\right) = \left\langle \frac{\partial^2 \chi}{\partial \phi^2}, n \right\rangle = -r.$$

(c) Then we have

$$n'\left(\frac{\partial \chi}{\partial \theta}\right) = \frac{\partial n}{\partial \theta} = (-\cos(\phi)\sin(\theta), \cos(\phi)\cos(\theta), 0) = -\left(\frac{\cos(\phi)}{R + r\cos(\theta)}\right)\frac{\partial \chi}{\partial \theta}$$

and

$$n'\left(\frac{\partial \chi}{\partial \phi}\right) = \frac{\partial n}{\partial \phi} = (-\sin(\phi)\cos(\theta), -\sin(\phi)\sin(\theta), \cos(\phi)) = \frac{1}{r}\frac{\partial \chi}{\partial \phi}.$$

So that

$$\Pi = \begin{pmatrix} -\frac{\cos(\phi)}{R + r\cos(\theta)} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}.$$

(d) Hence the principal curvatures are the diagonal entries in Π, the mean curvature H is one half the sum of the diagonal entries and the Gaussian curvature is $\frac{\cos(\phi)}{r(R + r\cos(\theta))}$.

6. Again the detailed calculations are on the web page. We summarise the results.
(a) P is the graph of the function $f(x, y) = (x^2 + y^2)$ so it has global parametrisation $\psi(x, y) = (x, y, x^2 + y^2)$. The tangent vectors are

$$\frac{\partial \psi}{\partial x} = (1, 0, 2x) \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = (1, 2y, 0)$$

and thus

$$\frac{\partial \psi}{\partial x} \times \frac{\partial \psi}{\partial y} = (-2x, -2y, 1)$$

and

$$n = \frac{1}{\sqrt{1 + 4x^2 + 4y^2}}(-2x, -2y, 1).$$

(b) Calculating gives

$$\alpha\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial x}\right) = \frac{2}{\sqrt{1 + 4x^2 + 4y^2}},$$

$$\alpha\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right) = 0$$

and

$$\alpha\left(\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial y}\right) = \frac{2}{\sqrt{1 + 4x^2 + 4y^2}}.$$

(c) Calculating gives

$$\Pi = \frac{2}{(1 + 4x^2 + 4y^2)^{3/2}} \begin{bmatrix} 1 + 4y^2 & -4xy \\ -4xy & 1 + 4x^2 \end{bmatrix}$$

(d) Using the fact that for any 2 by 2 matrix A the eigenvalues are the roots of $t^2 - \operatorname{tr}(A)t + \det(A)$ gives the principal curvatures as

$$\frac{2}{(1 + 4x^2 + 4y^2)^{1/2}} \qquad \text{and} \qquad \frac{2}{(1 + 4x^2 + 4y^2)^{3/2}}.$$

The mean curvature is

$$2 + 4x^2 + 4y^2$$

and the Gaussian curvature is

$$\frac{2}{(1 + 4x^2 + 4y^2)^2}.$$