Geometry of Surfaces III 2011

Assignment 5.

- Please hand up solutions to the starred questions for marking either in the lecture on Tuesday 18th October or in the Hand-In Box on Level 6 by 5.00 pm on that same day.

- I have included some questions about the second fundamental form which we might not get to until the first week back after the break.

1*. Let $\Sigma \subset \mathbb{R}^3$ be a surface with a parametrisation $\psi\colon U \to \Sigma$ for some U open in $\mathbb{R}^2$. Let $f\colon \Sigma \to \mathbb{R}$ be a smooth function as defined in Definition 3.10 and discussed in the subsequent Propositions. Recall that if $s \in \Sigma$ then $f'(s)\colon T_s\Sigma \to \mathbb{R}$. Assume $\psi(x) = s$ and show that

$$f'(s)\left(\frac{\partial \psi}{\partial x^i}(x)\right) = \frac{\partial f \circ \psi}{\partial x^i}(x).$$

2. Consider the set

$$C = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \quad \text{and} \quad x \neq 1\}$$

(a) Assume that $\psi(t) = (\cos(t), \sin(t), 0)$ is parametrisation and show that it is an arc-length parametrisation.

(b) Find T, N and B on C and calculate the curvature and torsion.

3*. Consider the curve $\gamma(t) = (t^3, 3t^2, 6t)$. Compute T, N and B. Show that

$$\kappa = \frac{2}{3(t^2+2)^2} \quad \text{and} \quad \tau = \frac{-2}{3(t^2+2)^2}.$$

Remember that if $X\colon C \to \mathbb{R}^3$ then the arc-length derivative and the derivative in the γ parametrisation are related by

$$\dot{X} = X' \|\gamma'\|^{-1}.$$

It is probably easier to compute T, N and B and then read off κ and τ but you can also use the general formula we have for κ.

4. Consider the curve $\gamma(t) = (3t - t^3, 3t^2, 3t + t^3)$. Compute T, N and B and find κ and τ. See the comments attached to the question above.

5*. Recall the torus $T \subset \mathbb{R}^3$ defined by $F(x, y, z) = \left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 - r^2 = 0$ for $0 < r < R$ and with parameters:

$$\chi(\theta, \phi) = (\cos(\theta)(R + r\cos(\phi)), \sin(\theta)(R + r\cos(\phi)), r\sin(\phi)).$$

(a) Compute the *outward* unit normal $n = \hat{n}$. To make sure it is the outward normal check the value at some point.

(b) Calculate the second fundamental form α.

(c) Compute the map Π given by $\Pi = -n'$. That is

$$\Pi\left(\frac{\partial \chi}{\partial \theta}\right) = -\frac{\partial n}{\partial \theta} \qquad \text{and} \qquad \Pi\left(\frac{\partial \chi}{\partial \phi}\right) = -\frac{\partial n}{\partial \phi}.$$

(d) Find the principal curvatures, the mean curvature and the Gaussian curvature.

6. Consider the paraboloid $P \subseteq \mathbb{R}^3$ defined by $z = x^2 + y^2$.

(a) Calculate the unit normal n oriented so that at $(0,0,0)$ it is $(0,0,1)$.

(b) Calculate the second fundamental form α.

(c) Compute the map Π given by $\Pi = -n'$.

(d) Find the principal curvatures, the mean curvature and the Gaussian curvature.