• Please hand up solutions to the starred questions for marking either in the lecture on Tuesday 18th October or in the Hand-In Box on Level 6 by 5.00 pm on that same day.

• I have included some questions about the second fundamental form which we might not get to until the first week back after the break.

1*. Let $\Sigma \subset \mathbb{R}^3$ be a surface with a parametrisation $\psi: U \to \Sigma$ for some $U$ open in $\mathbb{R}^2$. Let $f: \Sigma \to \mathbb{R}$ be a smooth function as defined in Definition 3.10 and discussed in the subsequent Propositions. Recall that if $s \in \Sigma$ then $f'(s): T_s \Sigma \to \mathbb{R}$. Assume $\psi(x) = s$ and show that

$$f'(s) \left( \frac{\partial \psi}{\partial x^i}(x) \right) = \frac{\partial f \circ \psi}{\partial x^i}(x).$$

2. Consider the set $C = \{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 = 1 \text{ and } x \neq 1\}$

(a) Assume that $\psi(t) = (\cos(t), \sin(t), 0)$ is parametrisation and show that it is an arc-length parametrisation.

(b) Find $T$, $N$ and $B$ on $C$ and calculate the curvature and torsion.

3*. Consider the curve $\gamma(t) = (t^3, 3t^2, 6t)$. Compute $T$, $N$ and $B$. Show that

$$\kappa = \frac{2}{3(t^2 + 2)^2} \quad \text{and} \quad \tau = \frac{-2}{3(t^2 + 2)^2}.$$ 

Remember that if $X: C \to \mathbb{R}^3$ then the arc-length derivative and the derivative in the $y$ parametrisation are related by

$$\dot{X} = X'\|y'\|^{-1}.$$ 

It is probably easier to compute $T$, $N$ and $B$ and then read off $\kappa$ and $\tau$ but you can also use the general formula we have for $\kappa$.

4. Consider the curve $\gamma(t) = (3t - t^3, 3t^2, 3t + t^3)$. Compute $T$, $N$ and $B$ and find $\kappa$ and $\tau$. See the comments attached to the question above.

5*. Recall the torus $T \subset \mathbb{R}^3$ defined by $F(x, y, z) = \left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 - r^2 = 0$ for $0 < r < R$ and with parameters:

$$\chi(\theta, \phi) = (\cos(\theta)(R + r \cos(\phi)), \sin(\theta)(R + r \cos(\phi)), r \sin(\phi)).$$

(a) Compute the outward unit normal $n = \hat{n}$. To make sure it is the outward normal check the value at some point.

(b) Calculate the second fundamental form $\alpha$.

(c) Compute the map $\Pi$ given by $\Pi = -n'$. That is

$$\Pi \left( \frac{\partial \chi}{\partial \theta} \right) = \frac{\partial n}{\partial \theta} \quad \text{and} \quad \Pi \left( \frac{\partial \chi}{\partial \phi} \right) = \frac{\partial n}{\partial \phi}.$$
(d) Find the principal curvatures, the mean curvature and the Gaussian curvature.

6. Consider the paraboloid $P \subseteq \mathbb{R}^3$ defined by $z = x^2 + y^2$.

   (a) Calculate the unit normal $n$ oriented so that at $(0, 0, 0)$ it is $(0, 0, 1)$.
   
   (b) Calculate the second fundamental form $\alpha$.
   
   (c) Compute the map $\Pi$ given by $\Pi = -n'$.
   
   (d) Find the principal curvatures, the mean curvature and the Gaussian curvature.