Geometry of Surfaces 2011

Assignment 4 — Solutions

1. We have $G_t'(x, y, z) = (2x, 2y, -2z)$. This is onto as a linear map $\mathbb{R}^3 \to \mathbb{R}$ if and only if it is not zero. It is only zero when $(x, y, z) = (0, 0, 0)$. This point is on $G_t^{-1}(0)$ only when $0^2 + 0^2 - 0^2 - t = 0$ i.e. when $t = 0$. Hence G_t is the defining equation of a submanifold if $t \neq 0$ and is not the defining equation of a submanifold if $t = 0$. In the former case $Z_t = G_t^{-1}(0)$ has dimension $3 - 1 = 0$. Two cases are:

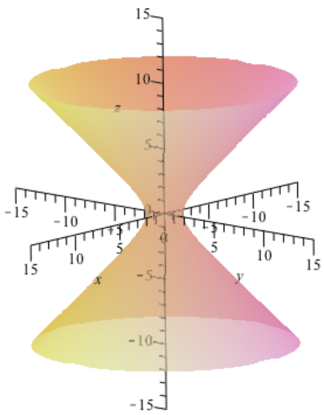


Figure 0.1: $t = 2$

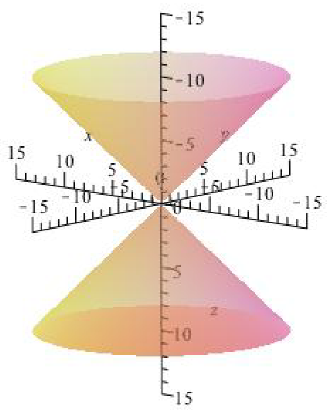


Figure 0.2: $t = 0$

2.

(a) We have $F'(x, y, z) = (2x, 2y, 0)$. This equals zero only when $x = 0$ and $y = 0$. But $F(0, 0, z) = -1 \neq 0$ so that $(x, y, 0) \notin F^{-1}(0)$. Hence $C = F^{-1}(0)$ is a submanifold of $\mathbb{R}^3$. It has dimension $3 - 1 = 2$.

(b) Standard facts about cos and sin tell us that ψ is one to one. Also

$$\psi'(\theta, t) = \begin{pmatrix} -\sin(\theta) & 0 \\ \cos(\theta) & 0 \\ 0 & 1 \end{pmatrix}$$

The columns are linearly independent unless $\sin(\theta) = \cos(\theta) = 0$ which is not possible. Hence $\psi(\theta, t)$ is one to one. Every point in C is in the image of θ except for those of the form $(1, 0, z)$ for $z \in \mathbb{R}$.

(c) From (b) we have

$$\frac{\partial \psi}{\partial \theta} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \psi}{\partial t} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

So that

$$\frac{\partial \psi}{\partial \theta}(\pi, 1) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \psi}{\partial t}(\pi, 1) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus the tangent space at $\psi(\pi, 1)$ is the span of these two vectors or the $Y - Z$ plane.

3*. As F_t has polynomial entries it is clearly smooth. Consider the derivative of F_t which is $F_t'(x, y, z)(a, b, c) = a(2x, 2x) + b(0, 2y) + c(2z, 0)$. At a point where $F_t(x, y, z) = 0$ we have that $x^2 + y^2 = 1$ and $x^2 + z^2 = t$. If $x = 0$ then y and z non-zero and $F_t'(x, y, z)$ is onto as $(0, 2y)$ and $(2z, 0)$ are linearly independent. So assume that $x \neq 0$. In that case as long as one of x or y is non-zero $F_t'(x, y, z)$ is onto. So assume that $y = z = 0$ and $x \neq 0$. Then $x^2 = 1$ and $x^2 = t$ so that $t = 1$. In that case $F_1(x, 0, 0)(a, b, c) = a(2x, 2x)$ is not onto. So F_t is certainly a defining equation if and only if $t \neq 0$.

The set Z_t is the intersection of a cylinder of radius 1 with a cylinder of radius t where the cylinders are at right angles to each other. For $t \neq 1$ the cylinders intersect in two disjoint circles. For $t = 1$ they intersect in two circles which themselves intersect in two points.

For $t \neq 1$ Z_t is a one-dimensional submanifold because the dimension of the kernel of F_t' is one-dimensional or because Z_t is defined as the zero set of $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $3 - 2 = 1$.

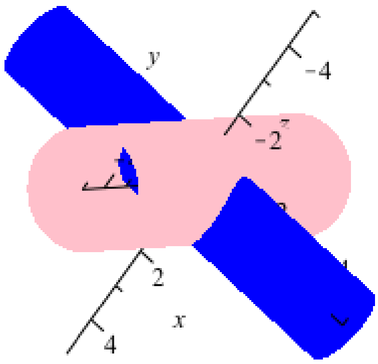


Figure 0.3: $t = 2$

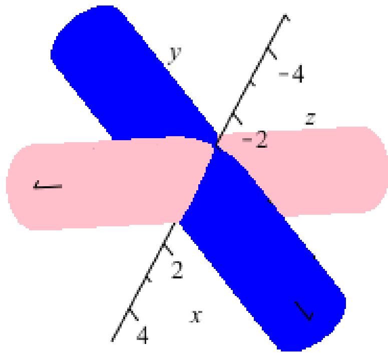


Figure 0.4: $t = 1$

4*.

(a) The derivatives of χ are

$$\frac{\partial \chi}{\partial \theta} = (-\sin(\theta)(R + r\cos(\phi)), \cos(\theta)(R + r\cos(\phi)), 0)$$

and

$$\frac{\partial \chi}{\partial \phi} = (-r\cos(\theta)\sin(\phi), -r\sin(\theta)\sin(\phi), r\cos(\phi))$$

and it is one to one if and only if these two vectors are linearly independent i.e. are not multiples of each other. Notice that we have

$$\left\langle \frac{\partial \chi}{\partial \theta}, \frac{\partial \chi}{\partial \phi} \right\rangle = 0, \quad \left\| \frac{\partial \chi}{\partial \theta} \right\|^2 = (R + r\cos(\theta))^2 > 0 \quad \text{and} \quad \left\| \frac{\partial \chi}{\partial \phi} \right\|^2 = r^2 > 0$$

as $R > r > 0$. Hence neither vector is zero and neither is a multiple of the other. So they are linearly independent. Hence χ is a parametrisation.

(b) Rather than sketch let me describe the set not in the image of χ. If $\theta = 0$ we get $\{(R + r\cos(\phi), 0, r\sin(\phi)) \mid \phi \in [0, 2\pi]\}$ that is a circle in the $X - Z$ plane centered at $(R, 0, 0)$ of radius r. If $\phi = 0$ then it is $\{((R + r)\cos(\theta), (R + r)\sin(\theta), 0) \mid \theta \in [0, 2\pi]\}$ that is a circle in the $X - Y$ plane centered at the origin of radius $R + r$. These intersect at $(R + r, 0, 0)$.

$\chi(\pi, \pi) = ((-1)(R - r), 0, 0) = (r - R, 0, 0)$ and $\chi(\pi/2, 3\pi/2) = (0, 1(R + 0), r(-1)) = (0, R, -r)$.

(c) $\frac{\partial \chi}{\partial \theta}(\pi, \pi) = (0, r - R, 0)$, $\frac{\partial \chi}{\partial \phi}(\pi, \pi) = (0, 0, -r)$ and the tangent space at $(r - R, 0, 0)$ is the $y - z$ plane. $\frac{\partial \chi}{\partial \theta}(\pi/2, 3\pi/2) = (-R, 0, 0)$, $\frac{\partial \chi}{\partial \phi}(\pi/2, 3\pi/2) = (0, r, 0)$ so the tangent space at $(0, R, -r)$ is the $x - y$ plane.

5*.

(a) Define $F\colon \mathbb{R}^3 \to \mathbb{R}$ by $F(x, y, z) = x^2 - y^2 - z$. Then $F'(x, y, z) = (2x, -2y, 1)$ which never vanishes. So $S = F^{-1}(0)$ is a submanifold.

(b)

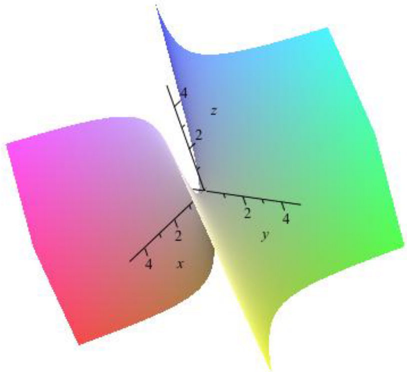
(c) We can parametrise S by $\psi(x, y) = (x, y, x^2 - y^2)$ and then

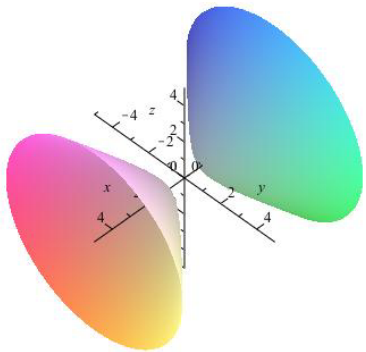
$$\frac{\partial \psi}{\partial x} = (1, 0, 2x) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = (1, 0, -2y).$$

and the tangent space is the span of these vectors.

(d) A general point has the form $(x, y, x^2 - y^2) + \alpha(1, 0, 2x) + \beta(1, 0, -2y)$ and applying F gives $\alpha^2 - \beta^2 = 0$. So this in S if and only if $\alpha = \pm\beta$. In each of these cases the whole line is in S.

6.





(a) Let $F(x, y, z) = x^2+y^2-z^2-1$. We have $F'(x, y, z) = (2x, 2y, -2z)$. This is zero if and only if $x = y = z = 0$ which is not on the curve.

(b)

(c) From $F'(x, y, z)$ we see that the tangent space is all (α, β, γ) such that $x\alpha + y\beta - z\gamma = 0$.

(d) Have a look at the hand written solutions I emailed around after the tutorial which are on the web page as well.

$7^{\#}$.

(a) Let B be a symmetric matrix and note that $XX^t = 1$. Then $B = (1/2)B+(1/2)B^t = (1/2)BXX^t+(1/2)XX^tB^t = ((1/2)BX)X^t + X((1/2)BX)^t$ so $\chi(A) = B$ if $A = (1/2)BX$. Hence χ is onto.

(b) Examination of the entries of $F(X)$ shows that they are polynomial in the entries of X and hence F is a smooth function. Notice that $F(X)^t = (XX^t)^t - I^t = XX^t - I = F(X)$ so that F has image in S_n.

(c) Notice that you cannot just launch off and compute you have to say that F is differentiable so therefore $F'(X)(A)$ is obtained by differentiating $F(X + tA) = XX^t + t(AX^t + XA^t) + t^2AA^t)$. This gives $F'(X)(A) = AX^t + XA^t$.

(d) $F'(X)$ is onto S_n for all $X \in F^{-1}(0) = O_n$. Hence O_n is a submanifold. Its dimension is $n^2 - \dim(S_n) = n^2 - n(n + 1)/2 = (1/2)n(n - 1)$.

(e) $T_1O_n = \ker F(1) = \{A \mid A^t + A = 0\}$. This is the set of all skew-symmetric matrices.