Geometry of Surfaces III 2011

Assignment 4.

- Please hand up solutions to the starred questions for marking either in the lecture on Tuesday 20th September or in the Hand-In Box on Level 6 by 5.00 pm on that same day.

- Honours students should also hand in the starred questions and the hashed question.

1. For each $t \in \mathbb{R}$ define the function $G_t \colon \mathbb{R}^3 \to \mathbb{R}$ by $G_t(x, y, z) = (x^2 + y^2 - z^2 - t)$. Find the t for which G_t is a defining equation for a submanifold and the t for which it is not. Let $Z_t \subset \mathbb{R}^3$ be defined by $G_t(x, y, z) = 0$. Sketch Z_t for a case when G_t is a defining equation and a case when it is not. What dimension is Z_t when it is a submanifold ?

2. Consider the cylinder C defined by the equation $F(x, y, z) = x^2 + y^2 - 1 = 0$.

(a) Show that C is a submanifold of $\mathbb{R}^3$. What is the dimension of C?

(b) Define $\psi \colon (0, 2\pi) \times \mathbb{R} \to C$ by $\psi(\theta, t) = (\cos(\theta), \sin(\theta), t)$. Show that ψ is a local parametrisation for C. What is missing from the image of ψ?.

(c) Calculate $\partial\psi/\partial\theta(\theta, t)$ and $\partial\psi/\partial t(\theta, t)$ and evaluate for $(\theta, t) = (\pi, 1)$. Describe the tangent space $T_{\psi(\pi,1)}C$ as subspace of $\mathbb{R}^3$.

3*. For each $t > 0$ define the function $F_t \colon \mathbb{R}^3 \to \mathbb{R}^2$ by $F_t(x, y, z) = (x^2 + z^2 - 1, x^2 + y^2 - t)$. Find the t for which this is a defining equation for a submanifold and the t for which it is not. Let $Z_t \subset \mathbb{R}^3$ be defined by $F_t(x, y, z) = (0, 0)$. Sketch Z_t for a case when F_t is a defining equation and a case when it is not. What dimension is Z_t when it is a submanifold ?

4*. Consider the torus $T \subset \mathbb{R}^3$ defined by $F(x, y, z) = \left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 - r^2 = 0$ for $0 < r < R$. We showed in lectures that this was a submanifold.

(a) Assuming that

$$\chi(\theta, \phi) = (\cos(\theta)(R + r\cos(\phi)), \sin(\theta)(R + r\cos(\phi)), r\sin(\phi))$$

is one to one show that it is a parametrisation of T for $(\theta, \phi) \in (0, 2\pi) \times (0, 2\pi)$.

(b) Sketch the part of T which is *not* in the image of χ and find the values of θ and ϕ for which $\chi(\theta, \phi) = (r - R, 0, 0)$ and $\chi(\theta, \phi) = (0, R, -r)$

(c) Find the vectors $\partial\chi/\partial\theta$ and $\partial\chi/\partial\phi$ at the points $(r - R, 0, 0)$ and $(0, R, -r)$, sketch them and describe the tangent space at these points.

5*. Consider the subset S of $\mathbb{R}^3$ defined by the equation $z = x^2 - y^2$.

(a) Show that S is a submanifold.

(b) Sketch or graph electronically the submaifold S.

(c) Calculate the tangent space at each point of S.

(d) Show that the set $s + T_s S$ intersects S in two straight lines going through S. Hint: write down the equation for a vector in $s + T_s S$ and see when it is on S.

Remark: S is called a hyperbolic paraboloid or a pringle and is an example of a double ruled surface. You can find it under its name on wikipedia.

6. Consider the subset S of $\mathbb{R}^3$ defined by the equation $z^2 + 1 = x^2 + y^2$.

(a) Show that S is a submanifold.

(b) Sketch or graph electronically the submanifold S.

(c) Calculate the tangent space at each point of S.

(d) Show that the set $s + T_s S$ intersects S in two straight lines going through S. Hint: write down the equation for a vector in $s + T_s S$ and see when it is on S. This seems to get a bit messier than the starred case above,.

Remark: S is called a hyperboloid of revolution and is another example of a double ruled surface. You can find it under its name on wikipedia.

$7^{\#}$. Consider M_n the vector space of all real n by n matrices and S_n the subspace of symmetric matrices. For this question you need to know that both these are isomorphic to $\mathbb{R}^N$ for suitable N. For example a matrix in M_n can be mapped to an n^2 vector by listing all the rows one after the other. Similarly we can identify the subspace of all symmetric matrices $S_n \subset M_n$ with $\mathbb{R}^m$ for suitable $m = n(n+1)/2$ by mapping the entries on and above the diagonal to a vector. Knowing this enables you to talk about smooth functions, apply the Inverse Function Theorem etc. However I suspect the questions are better done in terms of matrices rather than using an explicit identification of this kind. Recall the the O_n, the group of orthogonal matrices, is the subset of M_n for which $XX^t = 1$.

(a) Let $X \in O_n$. Show that the linear map $\chi\colon M_n \to S_n$ defined by $\chi(A) = AX^t + XA^t$ is onto.

(b) Show that the function $F(X) = XX^t - 1$ is a smooth function taking its values in S_n.

(c) Calculate the derivative $F'(X)\colon M_n \to S_n$ for any X in M_n. Hint: recall from lectures that once you know that F is differentiable you can compute $F'(X)(A)$ by differentiating $F(X + tA)$ with respect to t at 0.

(d) Show that O_n is a submanifold of M_n. What is its dimension ?

(e) Describe the tangent space to O_n at 1.

Remark: O_n is an example of a Lie group. It is a group and also a manifold for which the multiplication and inversion functions are smooth. Those of you who have done Topology and Analysis should be able to show it is compact and not connected (**not** part of assignment).