# Geometry of Surfaces 2011

## Assignment 3 — Solutions

1.

(a)

$$g \circ f(x^1, x^2) = (x^1x^2, (x^2)^3x^1 \sin(x^1), \exp((x^2)^2 \sin(x^1)))$$

(b)

$$f'(x^1, x^2) = \begin{pmatrix} (x^2)^2 \cos(x^1) & 2x^2 \sin(x^1) \\ x^2 & x^1 \end{pmatrix}$$

$$g'(x^1, x^2) = \begin{pmatrix} 0 & 1 \\ x^2 & x^1 \\ \exp(x^1) & 0 \end{pmatrix}$$

$$(g \circ f)'(x^1, x^2) = \begin{pmatrix} x^2 & x^1 \\ (x^2)^3(\sin(x^1) + x^1 \cos(x^1)) & 3(x^2)^2x^1 \sin(x^1) \\ (x^2)^2 \cos(x^1) \exp((x^2)^2 \sin(x^1)) & 2x^2 \sin(x^1) \exp((x^2)^2 \sin(x^1)) \end{pmatrix}$$

You have $f'(x^1, x^2) \colon \mathbb{R}^2 \to \mathbb{R}^2$, $g'(x^1, x^2) \colon \mathbb{R}^2 \to \mathbb{R}^3$ and $(g \circ f)'(x^1, x^2) \colon \mathbb{R}^2 \to \mathbb{R}^3$.

(c) You just need to calculate the product $g'(f(x^1, x^2))f'(x^1, x^2)$ and verify the result.

2*.

(a)

$$g \circ f(x^1, x^2, x^3) = (x^2x^3, (x^1)^2(x^2 \exp(x^2))x^3)$$

(b)

$$f'(x^1, x^2, x^3) = \begin{pmatrix} 2x^1 \exp(x^2) & (x^1)^2 \exp(x^2) & 0 \\ 0 & x^3 & x^2 \end{pmatrix}$$

$$g'(x^1, x^2) = \begin{pmatrix} 0 & 1 \\ x^2 & x^1 \end{pmatrix}$$

$$(g \circ f)'(x^1, x^2, X^3) = \begin{pmatrix} 0 & x^3 & x^2 \\ 2x^1x^2x^3 \exp(x^2) & (x^1)^2(x^3)(x^2 \exp(x^2) + \exp(x^2)) & (x^1)^2x^2 \exp(x^2) \end{pmatrix}$$

You have $f'(x^1, x^2, x^3) \colon \mathbb{R}^3 \to \mathbb{R}^2$, $g'(x^1, x^2) \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $(g \circ f)'(x^1, x^2, x^3) \colon \mathbb{R}^3 \to \mathbb{R}^2$.

(c) You just need to calculate the product $g'(f(x^1, x^2))f'(x^1, x^2)$ and verify the result.

3*.

(a) We have

$$f'(x^1, x^2) = \begin{pmatrix} 1 & 0 \\ (x^2)^2 & 2x^1x^2 \end{pmatrix}$$

so that $\det(f'(x^1, x^2)) = 2x^1x^2$. Hence $f'(x^1, x^2)$ is invertible unless $x^1 = 0$ or $x^2 = 0$.

(b) $f'(1, 1) = 2 \neq 0$ so $f'(1, 1)$ is invertible. It follows from the Inverse Function Theorem that there is an open set $U \subset \mathbb{R}^2$ containing $(1, 1)$ such that $f(U) \subset \mathbb{R}^2$ is open and $f_{|U} \colon U \to f(U)$ is a diffeomorphism.

(c) $f$ is not one to one as $f(1, 1) = (1, 1) = f(1, -1)$. $f$ is not onto as you cannot solve $(1, -1) = (x_1, x_1(x^2)^2) = f(x^1, x^2)$ as it is equivalent to $x^1 = 1$ and $(x^2)^2 = -1$.

4.

(a) We have

$$f'(x^1, x^2, x^3) = \begin{pmatrix} 1 & 0 & 0 \\ x^3 & 0 & x^1 \\ 1 & x^3 & x^2 \end{pmatrix}$$

so that $\det(f'(x^1, x^2, x^3) = -x^3x^1$ and hence $f'(x^1, x^2, x^3)$ is invertible unless $x^1 = 0$ or $x^2 = 0$.

(b) $f'(1,1,1) = 2 - 1 \neq 0$ so $f'(1,1,1)$ is invertible. It follows from the Inverse Function Theorem that there is an open set $U \subset \mathbb{R}^3$ containing $(1,1,1)$ such that $f(U) \subset \mathbb{R}^3$ is open and $f_{|U}\colon U \to f(U)$ is a diffeomorphism.

(c) $f$ is not one to one as $f(0,-1,-1) = (0,0,1) = f(0,1,1)$. $f$ is not onto as cannot solve $(x^1, x^1x^3, x^1 + x^2x^3) = (0,1,0)$.

5. Let $x \in \tilde{U}$ then $(x,0) \in U$ which is open so $\exists\epsilon > 0$ such that $B_{\mathbb{R}^{n+m}}((x,0),\epsilon) \subset U$. Let $y \in B_{\mathbb{R}^n}(x,\epsilon)$ then $\|(x,0) - (y,0)\|_{\mathbb{R}^{n+m}} = \|x - y\|_{\mathbb{R}^n} < \epsilon$ so that $(y,0) \in B_{\mathbb{R}^{n+m}}((x,0),\epsilon) \subset U$. Hence $(y,0) \in \tilde{U}$ so that $B_{\mathbb{R}^n}(x,\epsilon) \subset \tilde{U}$ and hence $\tilde{U}$ is open.

6*.

(a) Notice first that each $L(e^i)$ is in the image of $L$. Then let $y \in \{L(x) \mid x \in \mathbb{R}^n\}$. Then $y = L(x)$ for some $x \in \mathbb{R}^n$. But $x = \sum_{i=1}^n x^i e^i$ and hence

$$y = L(x) = L(\sum_{i=1}^n x^i e^i) = \sum_{i=1}^n x^i L(e^i)$$

because $L$ is linear. Thus the $L(e^i)$ span $\operatorname{im}(L)$.

(b) $(\Leftarrow)$ Assume the kernel of $L$ is zero. If we have $\sum_{i=1}^n x^i(e^i) = 0$ then $L(\sum_{i=1}^n x^i e^i) = 0$ so that $\sum_{i=1}^n x^i e^i = 0$. But the $e^i$ are a basis so $x^1 = x^2 = \cdots = x^n = 0$. Hence the $L(e^i)$ are a linearly independent. We already know from (a) that they span $\operatorname{im}(L)$ so they must be a basis. $(\Rightarrow)$. Assume the $L(e^i)$ are a basis so they are linearly independent. Let $L(x) = 0$. Then $x = \sum_{i=1}^n x^i e^i$ so that $0 = L(x) = \sum_{i=1}^n x^i L(e^i)$. But the $L(e^i)$ are linearly independent so $x^1 = x^2 = \ldots x^n = 0$ hence $x = \sum_{i=1}^n s x^i e^i = 0$. So $\ker(L) = 0$.

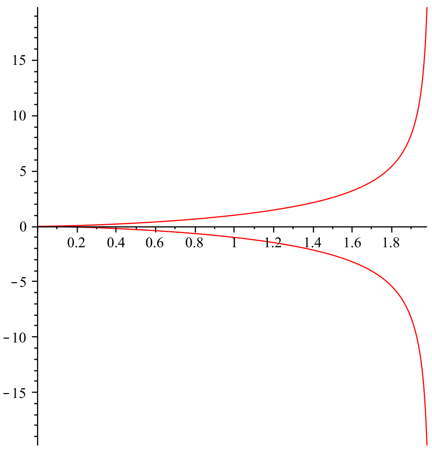
7.

(a) Let $y(t) = (y^1(t), y^2(t)$ and notice that $y^2(t) = ty^1(t)$. So if $y(s) = y(t)$ then $ty^1(t) = sy^1(t)$. If $y^1(t) = 0$ then $y^1(s) = 0$ and hence $s = t = 0$. If $y^1(t) \neq 0$ then $t = s$. So $y$ is one to one.

(b) $y'(t)\colon \mathbb{R} \to \mathbb{R}^2$ is one to one unless it is zero when it is not one to one. But

$$y'(t) = \left(\frac{4t}{(1+t^2)^2}, \frac{-t^3 + 6t^2}{(1+t^2)^2}\right)$$

which is zero only if $t = 0$. Hence the only point is $t = 0$.

(c) 

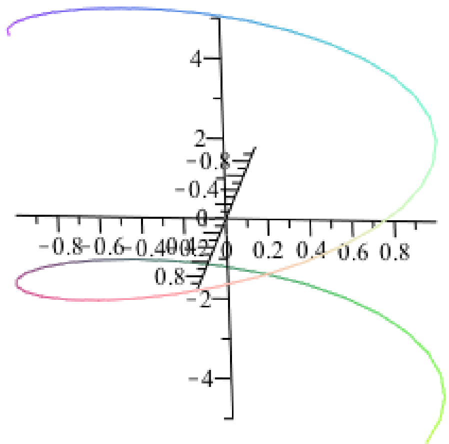

8*.

(a) As $y^3(t) = t$ clearly $y$ is one to one because if $y(t) = y(s)$ then $y^3(t) = y^3(s)$ so that $t = s$.

(b) $y'(t)\colon \mathbb{R} \to \mathbb{R}^3$ is one to one unless it is zero when it is not one to one. But

$$y'(t) = (\cos(t), -\sin(t), 1)$$

is never zero so there are no points at which $y'(t)$ is not one to one.




(c)