# Geometry of Surfaces III 2011

## Assignment 3.

- Please hand up solutions to the starred questions for marking either in the lecture on Tuesday 6th September or in the Hand-In Box on Level 6 by 5.00 pm on that same day.

- Honours students should also hand in question 5.

1. Define $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x^1, x^2) = (\sin(x^1)(x^2)^2, x^1x^2)$ and $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ by $g(x^1, x^2) = (x^2, x^1x^2, \exp(x^1))$.

(a) Compute $g \circ f\colon \mathbb{R}^2 \to \mathbb{R}^3$.

(b) Compute $f'(x)$, $g'(x)$ and $(g \circ f)'(x)$ and say what spaces they map between.

(c) Verify that $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$ by computing both sides of the equation explicitly.

2*. Define $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x^1, x^2, x^3) = (\exp(x^2)(x^1)^2, x^2x^3)$ and $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $g(x^1, x^2) = (x^2, x^1x^2)$.

(a) Compute $g \circ f\colon \mathbb{R}^3 \to \mathbb{R}^2$.

(b) Compute $f'(x)$, $g'(x)$ and $(g \circ f)'(x)$ and say what spaces they map between.

(c) Verify that $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$ by computing both sides of the equation explicitly.

3*. Consider the function $f(x^1, x^2) = (x^1, x^1(x^2)^2)$.

(a) At what points $x \in \mathbb{R}^2$ is $f'(x^1, x^2)$ invertible ?

(b) Use the Inverse Function Theorem to show that $f$ has a smooth inverse in an open set containing the point $(1, 1)$.

(c) Is $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ one to one ? Is $f$ onto ?

4. Consider the function $f(x^1, x^2, x^3) = (x^1, x^1x^3, x^1 + x^2x^3)$.

(a) At what points $x \in \mathbb{R}^3$ is $f'(x^1, x^2, x^3)$ invertible ?

(b) Use the Inverse Function Theorem to show that $f$ has a smooth inverse in an open set containing the point $(1, 1, 1)$.

(c) Is $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ one to one ? Is $f$ onto ?

5. Let $\widetilde{U} \subseteq \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ be open. Define

$$U = \{x \in \mathbb{R}^n \mid (x, 0) \in \widetilde{U}\} \subseteq \mathbb{R}^n$$

and show that $U$ is open.

6*. Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$. Denote by $e^i \in \mathbb{R}^n$ the usual basis vector which has a 1 in the $i$th position and zeros elsewhere.

(a) Show that the image of $L$ is spanned by the collection of vectors $L(e^i)$ for $i = 1, \dots, n$.

(b) Show that the vectors $L(e^i)$ are a basis for the image of $L$ if and only if the kernel of $L$ is zero.

Recall that

$$\ker(L) = \{x \in \mathbb{R}^n \mid L(x) = 0\} \subseteq \mathbb{R}^n$$

and

$$\operatorname{im}(L) = \{L(x) \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

7. Consider $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ defined by

$$\gamma(t) = \left(\frac{2t^2}{1+t^2}, \frac{2t^3}{1+t^2}\right)$$

(a) Is $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ one to one ?

(b) Find the points $\xi \in \mathbb{R}$ for which $\gamma'(\xi)\colon \mathbb{R} \to \mathbb{R}^2$ is not one to one.

(c) Sketch $\gamma$. You may use electronic assistance.

8*. Define $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ by

$$\gamma(t) = (\sin(t), \cos(t), t).$$

(a) Is $\gamma$ one to one ?

(b) Find the points $\xi \in \mathbb{R}$ for which $\gamma'(\xi)\colon \mathbb{R} \to \mathbb{R}^3$ is not one to one.

(c) Sketch $\gamma$. You may use electronic assistance.