Geometry of Surfaces 2011

Assignment 2 — Solutions

1*. We have $f(x, y, z) = (xy, y^2 + z)$. Let $h = (\alpha, \beta, \gamma)$ then

$$\begin{aligned} f(1+\alpha, 1+\beta, \gamma) &= ((1+\alpha)(1+\beta), (1+\beta)^2 + \gamma) \\ &= f(1,1,0) + (\alpha + \beta, 2\beta + \gamma) + (\alpha\beta, \beta^2) \end{aligned}$$

Define $f'(1,1,0)(\alpha, \beta, \gamma) = (\alpha + \beta, 2\beta + \gamma)$ and note that it is linear. Then

$$\begin{aligned} 0 \leq \frac{\| f(1+\alpha, 1+\beta, \gamma) - f(1,1,0) - f'(1,1,0)(\alpha,\beta,\gamma) \|}{\|h\|} &= \sqrt{\frac{\alpha^2\beta^2 + \beta^4}{\alpha^2 + \beta^2 + \gamma^2}} \\ &\leq \sqrt{\frac{(\alpha^2 + \beta^2 + \gamma^2)^2}{\alpha^2 + \beta^2 + \gamma^2}} \\ &\leq \|h\| \end{aligned}$$

which $\to 0$ as $\|h\| \to 0$ so the derivative exists and is the linear map $f'(1,1,0)(\alpha, \beta, \gamma) = (\alpha + \beta, 2\beta + \gamma)$. As a matrix this is

$$f'(1,1,0)(\alpha,\beta,\gamma) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

which agrees with the Jacobian matrix of partial derivatives

$$\begin{pmatrix} \frac{\partial f^1}{\partial x} & \frac{\partial f^1}{\partial y} & \frac{\partial f^1}{\partial z} \\ \frac{\partial f^2}{\partial x} & \frac{\partial f^2}{\partial y} & \frac{\partial f^2}{\partial z} \end{pmatrix} = \begin{pmatrix} y & x & 0 \\ 0 & 2y & 1 \end{pmatrix}$$

evaluated at $(x, y, z) = (1, 1, 0)$.

2. We have $f(x, y, z) = (x + y^2, yz)$. Let $h = (\alpha, \beta, \gamma)$ then

$$\begin{aligned} f(1+\alpha, 1+\beta, 1+\gamma) &= ((1+\alpha) + (1+\beta)^2, (1+\beta)(1+\gamma)) \\ &= f(1,1,1) + (\alpha + 2\beta, \beta + \gamma) + (\beta^2, \gamma\beta) \end{aligned}$$

Define $f'(1,1,1)(\alpha, \beta, \gamma) = (\alpha + 2\beta, \beta + \gamma)$ and note that it is linear. Then

$$\begin{aligned} 0 \leq \frac{\| f(1+\alpha, 1+\beta, 1+\gamma) - f(1,1,1) - f'(1,1,1)(\alpha,\beta,\gamma) \|}{\|h\|} &= \sqrt{\frac{\beta^4 + \beta^2\gamma^2}{\alpha^2 + \beta^2 + \gamma^2}} \\ &\leq \sqrt{\frac{(\alpha^2 + \beta^2 + \gamma^2)^2}{\alpha^2 + \beta^2 + \gamma^2}} \\ &\leq \|h\| \end{aligned}$$

which $\to 0$ as $\|h\| \to 0$ so the derivative exists and is the linear map $f'(1,1,1)(\alpha, \beta, \gamma) = (\alpha + 2\beta, \beta + \gamma)$. As a matrix this is

$$f'(1,1,1)(\alpha,\beta,\gamma) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

which agrees with the Jacobian matrix of partial derivatives

$$\begin{pmatrix} \frac{\partial f^1}{\partial x} & \frac{\partial f^1}{\partial y} & \frac{\partial f^1}{\partial z} \\ \frac{\partial f^2}{\partial x} & \frac{\partial f^2}{\partial y} & \frac{\partial f^2}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 2y & 0 \\ 0 & z & y \end{pmatrix}$$

evaluated at $(x, y, z) = (1, 1, 1)$.

3. From first principles we have

$$F(a+h) - F(a) = \|a+h\|^2 - \|a\|^2 = \|a\|^2 + 2\langle a, h \rangle + \|h\|^2 - \|a\|^2 = 2\langle a, h \rangle + \|h\|^2.$$

So let $F'(a)$ be the linear map $F'(a)(h) = 2\langle a, h\rangle$. Then

$$0 \le \frac{|F(a+h) - F(a) - F'(a)(h)|}{\|h\|} = \|h\|$$

which $\to 0$ as $\|h\| \to 0$. So F is differentiable at a and $F'(a)(h) = 2\langle a, h\rangle$.

The alternative approach is to note that $F(x) = (x^1)^2 + \cdots + (x^n)^2$ so that $\partial F/\partial x^i(a) = 2a^i$. Clearly the partial derivatives exist and are continuous for all $a \in \mathbb{R}^n$ so from Proposition 2.14 F is differentiable at all a and the Jacobian matrix $J(F)(a)$ applied to h is $2a^1h^1 + \cdots + 2a^nh^n = 2\langle a, h\rangle$ so $F'(a)(h) = 2\langle a, h\rangle$.

4*. Notice that $\phi = F \circ f$ where F is defined in the previous question. Also from that same question F is differentiable everywhere. So by the Chain Rule ϕ is differentiable everywhere. Moreover $\phi'(a) = F'(f(a)) \circ f'(a)$ thus $\phi'(a)(h) = F'(f(a))(f'(a)) = 2\langle f(a), f'(a)(h)\rangle$.

5*. We have

$$f(a+h) - f(a) - f'(a)(h) = L(a+h) + v - (L(a) + v) - L(h) = L(a) + L(h) + v - L(a) - v - L(h) = 0.$$

Thus

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - f'(a)(h)\|}{\|h\|} = \lim_{h\to 0} \frac{0}{\|h\|} = 0$$

so the derivative exists.

6. Note that f differs from F in question 3 by a constant so that $f'(w)(h) = F'(w)(h) = 2\langle w, h\rangle$. If $w = 0$ then $f'(w)(h) = 2\langle 0, h\rangle = 0$ so $f'(0)\colon \mathbb{R}^3 \to \mathbb{R}$ is not onto. Conversely if $w \neq 0$ then $f'(w)(\lambda w) = 2\lambda\|w\|^2 \neq 0$. Hence if $\rho \in \mathbb{R}$ we can choose λ so that $2\lambda\|w\|^2 = \rho$ so that $f'(w)(\lambda w) = \rho$ and thus $f'(w)$ is onto. The result about the kernel follows from the definition of $f'(w)$. That is $f'(w)(v) = 0$ if and only if $\langle w, v\rangle = 0$.

7. The first part is a straightforward calculation:

$$\left(\frac{2x^1}{1+\|x\|^2}\right)^2 + \left(\frac{2x^2}{1+\|x\|^2}\right)^2 + \left(\frac{1-\|x\|^2}{1+\|x\|^2}\right)^2 = \frac{4\|x\|^2 + 1 - 2\|x\|^2 + \|x\|^4}{(1+\|x\|^2)^2} = \frac{(1+\|x\|^2)^2}{(1+\|x\|^2)^2} = 1.$$

The Jacobian matrix is

$$J(\phi)(x) = \frac{1}{(1+\|x\|^2)^2}\begin{pmatrix} 2 + 2(x^2)^2 - 2(x^1)^2 & -4x^1x^2 \\ -4x^1x^2 & 2 + 2(x^1)^2 - 2(x^2)^2 \\ -4x^1 & -4x^2 \end{pmatrix}$$

Clearly the partial derivatives exist and are continuous so that ϕ is C^1. Hence ϕ is differentiable and the derivative $\phi'(x)$ is given by multiplication by the Jacobian matrix.

Notice that $\operatorname{im}(\phi'(x))$ is the span of the columns of $J(\phi)(x)$. Hence it is at most 2-dimensional. If the first column is zero for some x^1 and x^2 then $x^1 = 0$ and $2 + 2(x^2)^2 = 0$ which is not possible. Similarly for the second column. So both columns are always non-zero. A calculation shows that the columns are orthogonal. It follows that the columns are linearly independent and thus form a basis for $\operatorname{im}(\phi'(x))$. Hence $\operatorname{im}(\phi'(x))$ is 2-dimensional.

Another calculation shows that $\langle \phi(x), \phi'(x)(h)\rangle = 0$ for all h. Hence $\operatorname{im}(\phi'(x)) \subset \phi(x)^\perp$. Alternatively notice that $G(x) = \|\phi(x)\|^2 = 1$ for all x so by the Chain Rule and question 4 we see that $0 = G'(x)(h) = 2\langle \phi(x), \phi'(x)(h)\rangle$ for all h. But $\|\phi(x)\| = 1$ so $\phi(x) \neq 0$ and thus $\dim \phi(x)^\perp = 2$. Hence $\operatorname{im}(\phi'(x)) = \phi(x)^\perp$.

8*. The first part is a straightforward calculation that

$$(\cos(x^1)\sin(x^2))^2 + (\sin(x^1)\sin(x^2))^2 + (\cos(x^2))^2 = 1.$$

The Jacobian matrix is

$$J(\phi)(x) = \begin{pmatrix} -\sin(x^1)\sin(x^2) & \cos(x^1)\cos(x^2) \\ \cos(x^1)\sin(x^2) & \sin(x^1)\cos(x^2) \\ 0 & -\sin(x^2) \end{pmatrix}$$

Clearly the partial derivatives exist and are continuous so that ϕ is C^1. Hence ϕ is differentiable and the derivative $\phi'(x)$ is given by multiplication by the Jacobian matrix.

Notice that $\operatorname{im}(\phi'(x))$ is the span of the columns of $J(\phi)(x)$. Hence it is at most 2-dimensional. The norm of the first column is $\sin^2(x^2)$ which cannot vanish for $x^2 \in (0, \pi)$ so that the first column is never zero. The norm of the length of the second column is 1 so it cannot vanish either. So both columns are always non-zero. A calculation shows that the columns are orthogonal. It follows that the columns are linearly independent and thus form a basis for $\operatorname{im}(\phi'(x))$. Hence $\operatorname{im}(\phi'(x))$ is 2-dimensional.

Another calculation shows that $\langle \phi(x), \phi'(x)(h) \rangle = 0$ for all h. Hence $\text{im}(\phi'(x)) \subset \phi(x)^\perp$. Again this can be done by the chain rule as well. But $\|\phi(x)\| = 1$ so $\phi(x) \neq 0$ and thus $\dim \phi(x)^\perp = 2$. Hence $\text{im}(\phi'(x)) = \phi(x)^\perp$.

9*. First notice that f is continuous on all of $\mathbb{R}$ except possibly at zero. At zero we have $\lim_{x \to 0} f(x) = 0$ as $\sin(1/x)$ is bounded and $x^2 \to 0$ as $x \to 0$. Hence f is continuous or C^0 on all of $\mathbb{R}$. Clearly f is differentiable away from zero and the derivative is

$$f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$$

for $x \neq 0$. This doesn't have a limit as $x \to 0$ because the $\cos(1/x)$ oscillates between -1 and 1 so it is not possible that f is C^1. If we check the derivative at 0 we have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \left(x \sin(\frac{1}{x}) - \frac{0}{x} \right) = 0.$$

so $f'(0) = 0$. Hence f is differentiable on all of $\mathbb{R}$.

10. If we let $t = 1/(1 - x^2)$ then as $x \to \pm 1$ we have $t \to \infty$ so that the given inequality shows that $f(x) \to 0$ as $x \to \pm 1$. So $f(x)$ is continuous on $\mathbb{R}$. The derivative of f is

$$f(x) = \begin{cases} \exp(\frac{-1}{1-x^2}) \frac{-2x(1-x^2)^{k-1} + (1-x^2)p'(x) + 2kp(x)}{(1-x^2)^{k+1}} & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

which is of the same form. The limits of the derivatives from left and right at ± 1 are zero so by the Lemma in class f is C^1. But f' is of the same form as f so f' is C^1 and hence f is C^2. By iteration f is smooth.

It follows that h is smooth. Letting

$$g(x) = \frac{\int_{-\infty}^{x} h(s)ds}{\int_{-\infty}^{\infty} h(s)ds}$$

gives the required result. Some scaling like

$$\phi(x) = g\left(\frac{2\|x\|^2 - \delta^2 - \epsilon^2}{\delta^2 - \epsilon^2} \right)$$

should give the required result.