Geometry of Surfaces III 2011

Assignment 2.

- Please hand up solutions to the starred questions for marking either in the lecture on Tuesday 23rd August or in the Hand-In Box on Level 6 by 5.00 pm on that same day.

- Honours students please hand in the starred questions and question 10.

- For questions involving partial derivatives I am happy for you to make any reasonable claim about functions being continuous or differentiable that you might know from another course. But if I ask for you to prove differentiability from first principles then you need to calculate the limit.

1*. Define $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x,y,z) = (xy, y^2 + z)$. Find the derivative $f'(1,1,0)\colon \mathbb{R}^3 \to \mathbb{R}^2$ from first principles (i.e compute the limit) and verify that it equals the Jacobian matrix by also computing the partial derivatives.

2. Define $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x,y,z) = (x + y^2, zy)$. Find the derivative $f'(1,1,1)\colon \mathbb{R}^3 \to \mathbb{R}^2$ from first principles (i.e compute the limit) and verify that it equals the Jacobian matrix by also computing the partial derivatives.

3. Consider the function $F\colon \mathbb{R}^n \to \mathbb{R}$ defined by $F(x) = \|x\|^2$. Show that F is differentiable and that $F'(a)(h) = 2\langle a, h\rangle$ for all $a \in \mathbb{R}^n$. You can either use first principles or show that F is in $C^1(\mathbb{R}^n)$ by calculating the partial derivatives and then apply Proposition 2.14 from lectures.

4*. Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Define the function $\phi\colon \mathbb{R}^n \to \mathbb{R}$ by $\phi(x) = \|f(x)\|^2$. Show that if f is differentiable so also is ϕ and find a formula for $\phi'(x)$. You may use the previous question.

5*. Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear and $v \in \mathbb{R}^m$. Define $f\colon \mathbb{R}^n \to \mathbb{R}^m$ by $f(x) = L(x) + v$ and show from first principles that $f'(a) = L$ for all $a \in \mathbb{R}^n$.

6. Define $f\colon \mathbb{R}^3 \to \mathbb{R}$ by $f(w) = \|w\|^2 - 1$. For any $w = (x,y,z) \in \mathbb{R}^3$ calculate the derivative $f'(w)\colon \mathbb{R}^3 \to \mathbb{R}$ (you can use partial derivatives or question 3). Show that $f'(w)$ is onto as a linear map if and only if $w \neq (0,0,0)$. Show also that the kernel of $f'(u)$ is

$$\ker f'(u) = \{v \in \mathbb{R}^3 \mid \langle u, v\rangle = 0\}.$$

7. Define $\phi\colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\phi(x) = \left(\frac{2x^1}{1 + \|x\|^2}, \frac{2x^2}{1 + \|x\|^2}, \frac{1 - \|x\|^2}{1 + \|x\|^2}\right).$$

Show that the image of ϕ is inside

$$S^2 = \{u \in \mathbb{R}^3 \mid \|u\|^2 = 1\}.$$

Show that ϕ is C^1 and calculate $\phi'(x)\colon \mathbb{R}^2 \to \mathbb{R}^3$ for all x (you can use partial derivatives). Prove that the image of $\phi'(x)$ is two-dimensional and satisfies

$$\operatorname{im} \phi'(x) = \phi(x)^\perp = \{v \in \mathbb{R}^3 \mid \langle v, \phi(x)\rangle = 0\}.$$

8*. Let $U \subset \mathbb{R}^2$ be defined by $U = (0, 2\pi) \times (0, \pi)$ and define $\phi \colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\phi(x^1, x^2) = (\cos(x^1)\sin(x^2), \sin(x^1)\sin(x^2), \cos(x^2))$$

Show that the image of ϕ is inside

$$S^2 = \{u \in \mathbb{R}^3 \mid \|u\|^2 = 1\}.$$

Show that ϕ is C^1 and calculate $\phi'(x) \colon U \to \mathbb{R}^3$ for all x (you can use partial derivatives). Prove that the image of $\phi'(x)$ is two-dimensional and satisfies

$$\operatorname{im}\phi'(x) = \phi(x)^\perp = \{v \in \mathbb{R}^3 \mid \langle v, \phi(x)\rangle = 0\}.$$

9*. Consider the function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that f is C^0 on $\mathbb{R}$ and differentiable at all points of $\mathbb{R}$ (including 0) but not C^1 on $\mathbb{R}$. You will need to use first principles to get the derivative at 0.

Remark: It is worth plotting $f(x)$ and $f'(x)$ on a graphics calculator, or www.wolframalpha.com or some other plotting package. This is not required for the assessment.

10. Consider a function $f \colon \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = \begin{cases} \exp(\frac{-1}{1-x^2}) \frac{p(x)}{(1-x^2)^k} & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

for some polynomial p and natural number k. Use that fact that $\lim_{t\to\infty} \exp(-t)t^k = 0$ for any natural number k to deduce that f is continuous. Show that the derivative of f has the same form as f with a different k and p and hence deduce that f is smooth.

Consider now

$$h(x) = \begin{cases} \exp(\frac{-1}{1-x^2}) & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

By integrating h find a smooth function $g \colon \mathbb{R} \to \mathbb{R}$ with the property that $g(x)$ is zero for $x < -1$ and $g(x)$ is one for $x > 1$. Show that for any $\epsilon > \delta > 0$ there is a smooth function $\phi \colon \mathbb{R}^n \to \mathbb{R}$ with $\phi(x)$ equal to zero if $\|x\| > \epsilon$ and ϕ equal to one if $\|x\| < \delta$.

Remark: It is worth plotting $h(x)$ on a graphics calculator, or www.wolframalpha.com or some other plotting package. This is not required for the assessment.