Geometry of Surfaces 2011

Assignment 1 — Solutions

1. For Cauchy's inequality we note first that if $y = 0$ then the inequality holds as both sides are zero. So we can assume that $y \neq 0$. In that case we consider

$$\langle x - \frac{\langle x, y\rangle}{\langle y, y\rangle} y, x - \frac{\langle x, y\rangle}{\langle y, y\rangle} y\rangle \geq 0.$$

Expanding and simplifying gives Cauchy's inequality.

For the triangle inequality notice that from Cauchy's inequality we have that

$$\|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2.$$

But this is just

$$\langle x + y, x + y\rangle \leq (\|x\| + \|y\|)^2$$

and taking square roots preserves the inequality and gives the required result.

2*. Notice that from the triangle inequality $\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\|$. Swapping x and y gives $\|y\| \leq \|y - x\| + \|x\| = \|x - y\| + \|y\|$ as $\|-w\| = \|w\|$. Subtracting gives us $(\|x\| - \|y\|) \leq \|x - y\|$ and $(\|y\| - \|x\|) \leq \|x - y\|$. But $|\|x\| - \|y\||$ is either $\|x\| - \|y\|$ or $\|y\| - \|x\|$. This gives the required result.

3*. We have from (2)

$$0 \leq |\|x\| - \|x_n\|| \leq \|x - x_n\|$$

But as $x_n \to x$ we know that $\|x - x_n\| \to 0$ as $n \to \infty$. Hence by the Squeeze Lemma we conclude that $|\|x\| - \|x_n\| \to 0$ as $n \to \infty$. But this is the definition of $\lim_{n\to\infty} \|x_n\| = \|x\|$ so we are done.

4. We have

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x, y\rangle - \langle x, y_n\rangle + \langle x, y_n\rangle| \\ &= |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle| \\ &\leq |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle| \\ &\leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\|. \end{aligned}$$

Now from (3) we have that $\|y_n\| \to \|y\|$ and thus if we take $\epsilon = 1$ we have that there is an N such that for any $n \geq N$ $|\|y_n\| - \|y\|| \leq 1$ and thus $\|y_N\| \leq \|y\| + 1$. Hence

$$0 \leq |\langle x_n, y_n\rangle - \langle x, y\rangle| \leq \|x_n - x\|(1 + \|y\|) + \|x\|\|y_n - y\|.$$

The RHS goes to zero as $n \to \infty$ so by the Squeeze Lemma $|\langle x_n, y_n\rangle - \langle x, y\rangle| \to 0$ and thus $\langle x_n, y_n\rangle \to \langle x, y\rangle$ as $n \to \infty$.

5*. First we note that

$$\max\{|x^1|^2, \dots, |x^n|^2\} \leq \sum_{i=1}^{n} (x_i)^2$$

and that

$$\sqrt{\max\{|x^1|^2, \dots, |x^n|^2\}} = \max\{|x^1|, \dots, |x^n|\}.$$

So taking square roots of the first equation gives

$$\max\{|x^1|, \dots, |x^n|\} \leq \|x\|.$$

For the other side

$$\sum_{i=1}^{n} (x_i)^2 \leq \sum_{i=1}^{n} \max\{|x^1|^2, \dots, |x^n|^2\} = n \max\{|x^1|^2, \dots, |x^n|^2\}.$$

and taking square roots gives

$$\|x\| = \sqrt{n} \max\{|x^1|, \dots, |x^n|\}.$$

6. Consider $y \in B(x, \epsilon)$. We want to find a $\delta > 0$ such that $B(y, \delta) \subset B(x, \epsilon)$. It is best to draw a picture and make a guess for the radius of the circle around y that makes that ball sit inside the ϵ radius ball around x. A good guess is $\delta = \epsilon - \|x - y\|$. Notice that $\delta > 0$ as we required. Let $z \in B(y, \delta)$ so that $\|z - y\| < \epsilon - \|x - y\|$. Then

$$\|x - z\| \leq \|x - y\| + \|y - z\| < \epsilon$$

so that $z \in B(x, \epsilon)$. But z was any element of $B(y, \delta)$ so that $B(y, \delta) \subset B(x, \epsilon)$.

7. I'll talk about this in the tute and draw some pictures. The answers are

(a) For a tetrahedron we have $f = 4$, $e = 6$ and $v = 4$ so that $\chi = 4 - 6 + 4 = 2$.

(b) After you identify the edges of the square you should get $f = 2$, $e = 3$ and $v = 1$ so that $\chi = 2 - 3 + 1 = 0$.

(c) You should get $\chi = -2$.

(d) For a surface of genus g we have $\chi = 2 - 2g$. I'll try and do the drawing in the tutorial.